Tests of the L-Functions Ratios Conjecture.

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History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;
- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) = \sum_{m \le x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \le y} \frac{a_n}{n^{1-s}};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}} \omega_f \sum_{\gamma} g\left(\gamma \frac{\log N_f}{2\pi}\right) = \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots) g(\cdots)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha} R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

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where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Sympletic Results

Symplectic Families

- Fundamental discriminants: d square-free and 1 modulo 4, or $d/4$ square-free and 2 or 3 modulo 4.
- Associated character χ_d:
  ◇ $\chi_d(-1) = 1$ say d even;
  ◇ $\chi_d(-1) = -1$ say d odd.
  ◇ even (resp., odd) if $d > 0$ (resp., $d < 0$).

Will study following families:

◇ even fundamental discriminants at most X;
◇ $\{8d : 0 < d \le X, d$ an odd, positive square-free fundamental discriminant$\}$.

Prediction from Ratios Conjecture

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\pm\frac{i\pi\tau}{\log X}\right)\right]d\tau$$
$$+\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right.$$
$$\left.-e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau + O(X^{-\frac{1}{2}+\epsilon}),$$

with

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}$$
$$A_D'(r;r) = \sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Prediction from Ratios Conjecture

Main term is

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx$$
$$+\,O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\widehat{g}) \subset (-1,1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture

Assuming RH for $\zeta(s)$, for $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$:

$$\frac{-2}{X^* \log X} \sum_{d \le X} \int_{-\infty}^{\infty} g(\tau)\, e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau$$

$$= -\frac{g(0)}{2} + O(X^{-\frac{3}{4}(1-\sigma)+\epsilon});$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ error if $\sigma < 1/3$.

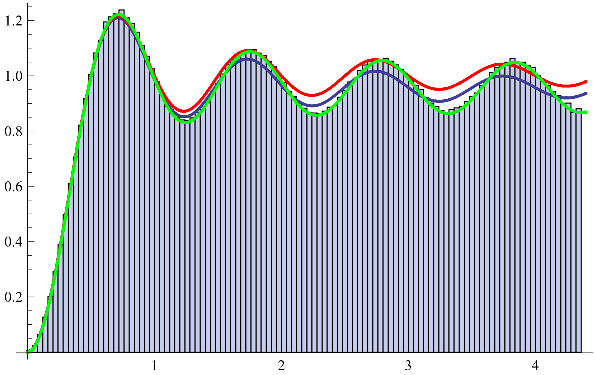
Main Results

Theorem (M– ’07)

Let $\operatorname{supp}(\widehat{g}) \subset (-\sigma, \sigma)$, *assume RH for* $\zeta(s)$. 1-*Level Density agrees with prediction from Ratios Conjecture*

- *up to* $O(X^{-(1-\sigma)/2+\epsilon})$ *for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X*;*
- *up to* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ *for our sub-family. If* $\sigma < 1/3$ *then agrees up to* $O(X^{-1/2+\epsilon})$.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^{6}]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).
$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.
$\diamond$ Green: all lower order terms.

Orthogonal Results

Background

Study $L(s,f)=\sum \lambda_f(n)n^{-s}$ with f ranging over cuspidal newforms of weight k and prime level $N\to\infty$.

Iwaniec-Luo-Sarnak calculated 1-level density if $\mathrm{supp}(\widehat{\phi})\subset(-2,2)$.

Key ingredient: averaging $\lambda_f(n)$'s over family by the Petersson formula.

Petersson Formula

Let

$$\Delta_{k,N}(m,n) \;=\; \sum_{f\in\mathcal{B}_k(N)} \omega_f(N)\lambda_f(m)\lambda_f(n).$$

We have

$$\Delta_{k,N}(m,n) \;=\; \delta(m,n) + 2\pi i^k \sum_{c\equiv 0 \text{ mod } N} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where $\delta(m,n)$ is the Kronecker symbol

$$S(m,n;c) \;=\; \sum_{d \text{ mod } c}{}^{*} \exp\left(2\pi i\frac{md+n\overline{d}}{c}\right)$$

is the classical Kloosterman sum ($d\overline{d} \equiv 1 \text{ mod } c$), and $J_{k-1}(x)$ is a Bessel function.

Consequences of the Petersson Formula

The Bessel-Kloosterman piece contributes an error term if $\sigma < 1$ and a main term otherwise.

The ‘diagonal’ piece does not include the Bessel-Kloosterman term, which we know contributes!

Possible danger: Ratios Conjecture says only to keep diagonal or main terms, and dropping a smaller ocntribution which becomes quite large!

Main Results: Test for family $\mathcal{F} = H_k^{\pm}(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

Theorem: Ratios Conjecture Prediction

With $\chi(s) = \prod_p \left(1 + \frac{1}{(p-1)p^s}\right)$, the 1-level density is

$$\sum_p \frac{2\log p}{p\log R}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)$$
$$\mp 2\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} X_L\left(\frac{1}{2}+2\pi i x\right)\chi(\epsilon+4\pi i x)\phi(t\log R)dt$$
$$-\int_{-\infty}^{\infty}\frac{X_L'}{X_L}\left(\frac{1}{2}+2\pi i t\right)\phi(t\log R)dt + O(N^{-1/2+\epsilon}),$$

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Theorem: Agreement with Number Theory

Assume GRH for $\zeta(s)$, Dirichlet L-functions, and $L(s, f)$. For ϕ such that $\text{supp}(\widehat{\phi}) \subset (-1, 1)$, the 1-level density agrees with the ratios conjecture prediction up to $O(N^{-1/2+\epsilon})$, and get agreement up to a power savings in N if $\text{supp}(\widehat{\phi}) \subset (-2, 2)$.

Sketch of Symplectic Proofs

Ratios Calculation

Hardest piece to analyze is

$$R(g;X) \;=\; -\frac{2}{X^* \log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)e^{-2\pi i\tau\frac{\log(d/\pi)}{\log X}}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)} \cdot \zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)d\tau,$$

$$A_D(-r,r) \;=\; \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of Γ and zeta functions.

Ratios Calculation: Weaker result for $\mathrm{supp}(\widehat{g}) \subset (-1,1)$.

- d-sum is $X^* e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} \left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} + O(X^{1/2})$;

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- decay of g restricts τ-sum to $|\tau| \le \log X$, Taylor expand everything but g: small error term and

$$\int_{|\tau|\le\log X} g(\tau) \sum_{n=-1}^{N} \frac{a_n}{\log^n X}(2\pi i\tau)^n e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau$$

$$= \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\le\log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;$$

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$$= \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\le \log X} (2\pi i \tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;$$

- from decay of g can extend the τ-integral to $\mathbb{R}$ (essential that N is fixed and finite!), for $n \ge 0$ get the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1 - \frac{\pi}{\log X}$, vanishes if $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$.

Number Theory Sums

$$\begin{aligned} S_{\mathrm{even}} &= -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p}\frac{\chi_d(p)^2\log p}{p^{\ell}\log X}\,\widehat{g}\left(2\frac{\log p^{\ell}}{\log X}\right) \\ S_{\mathrm{odd}} &= -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right). \end{aligned}$$

Number Theory Sums

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$*. Then*

$$\begin{aligned} S_{\rm even} &= -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i\tau}{\log X}\right)d\tau \\ &\quad + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)A_D'\left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\ S_{\rm odd} &= O(X^{-\frac{1-\sigma}{2}}\log^6 X). \end{aligned}$$

If instead we consider the family of characters χ_{8d} *for odd, positive square-free* $d \in (0, X)$ *(*d *a fundamental discriminant), then*

$$S_{\rm odd} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

Analysis of $S_{\rm even}$

$\chi_d(p)^2 = 1$ except when $p|d$. Replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$:

$$\begin{aligned} S_{\rm even} &= -2\sum_{\ell=1}^{\infty}\sum_{p}\frac{\log p}{p^{\ell}\log X}\,\widehat{g}\left(2\frac{\log p^{\ell}}{\log X}\right) \\ &\quad +\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p|d}\frac{\log p}{p^{\ell}\log X}\,\widehat{g}\left(2\frac{\log p^{\ell}}{\log X}\right) \\ &= S_{\rm even;1}+S_{\rm even;2}. \end{aligned}$$

Lemma (Perron's Formula)

$$S_{\rm even;1} = -\frac{g(0)}{2}+\frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)d\tau.$$

Analysis of S_{even}: $S_{\text{even};2}$

This piece gives us $\int g(\tau)A_D'(-\cdots,\cdots)$.

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- Main ideas:
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  - ⋄ For $p < X^{1/2}$: $\sum_{d\le X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.

Analysis of S_{even}: $S_{\text{even};2}$

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- Main ideas:
  - ◇ Restrict to $p \le X^{1/2}$.
  - ◇ For $p < X^{1/2}$: $\sum_{d\le X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.
  - ◇ Use Fourier Transform to expand $\widehat{g}$.

Analysis of S_{odd}

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right)\sum_{d\le X}\chi_d(p).$$

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Jutila's bound

$$\sum_{\substack{1<n\le N \\ n \text{ non-square}}}\left|\sum_{\substack{0<d\le X \\ d \text{ fund. disc.}}}\chi_d(n)\right|^2 \ll NX\log^{10}N.$$

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Proof: Cauchy-Schwarz and Jutila: $p^{2\ell+1}$ non-square:

$$\left(\sum_{\ell=0}^{\infty}\sum_{p^{(2\ell+1)/2}\le X^\sigma}\left|\sum_{d\le X}\chi_d(p)\right|^2\right)^{1/2} \ll X^{\frac{1+\sigma}{2}}\log^5 X.$$

Analysis of S_{odd}: Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums.

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$*. For family* $\{8d \,:\, 0 < d \le X,\ d$ *an odd, positive square-free fundamental discriminant*$\}$*,* $S_{\mathrm{odd}} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$*. In particular, if* $\sigma < 1/3$ *then* $S_{\mathrm{odd}} = O(X^{-1/2+\epsilon})$*.*

Conclusions

Conclusions

- Ratios Conjecture gives detailed predictions (up to $X^{1/2+\epsilon}$).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.

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