Title : *Overdetermined PDE's in Riemannian Geometry*

Abstract : An overdetermined problem gives rise to the following question : is it possible to identify, or characterize, the geometry of a compact domain Ω in a Riemannian manifold assuming the existence of a solution u of a certain PDE such that both u and its normal derivative are constant on the boundary of Ω ?

As a source of inspiration, we will start from the famous Pompeiu problem, an outstanding open problem in integral geometry, in the PDE formulation given by Williams in terms of the Schiffer conjecture. We then discuss in details the Serrin problem, first in Euclidean space (by giving two independent proofs of his celebrated rigidity result) and then in the other constant curvature space forms.

Then, we discuss isoparametric hypersurfaces in the sphere, which give rise to interesting solutions of the Serrin problem which are not isometric to geodesic balls. The study of the so-called "isoparametric tubes", in the general setting, will provide a large class of manifolds supporting non-trivial solutions to overdetermined PDE, and will serve as a kind of model space for these kind of problems.

Finally, we will give a quick look at the heat equation, and study an overdetermined problem for the integral of the heat kernel with Dirichlet boundary conditions. In that case, it is possible to show that the family of (compact) manifolds supporting solutions to the problem coincides with the family of isoparametric tubes. This gives a new interpretation of the isoparametric property in terms of heat diffusion. Other topics might include the following. Exotic solutions to the Serrin problem : minimal free boundary immersions. Overdetermined problem for the Steklov problem.

Prerequisities are just knowledge of basic facts on analysis on Riemannian manifolds.

Rather than being exhaustive on the (huge) literature on the subject, the main scope here is to give a geometric, more than analytic, flavor to the exposition, by stressing in particular the role of mean curvature in the construction of the examples.

Proofs will be given to illustrate the problem, and will try to avoid technicalities as much as possible; the role of the maximum principle is stressed, together with the Bochner formula.

Short syllabus.

Lecture 1. Definition of Pompeiu problem. The ball does not have the Pompeiu property. Pompeiu problem and the Schiffer conjecture. General form of the overdetermined PDE. Serrin problem in Euclidean space: proof by moving plane method (maximum principle + reflection method) , Weinberger proof (maximum principle + Bochner formula). Serrin problem in hyperbolic space and in the hemisphere (without proof).

Lecture 2. Serrin problem in the whole sphere : isoparametric boundaries.

Clifford tori are harmonic.

Review of second fundamental form and mean curvature. Isoparametric hypersurfaces: Cartan definition. Cartan polynomials and main facts. Isoparametric tubes in general Riemannian manifolds. Averaging operator. Isoparametric tubes have the harmonic property.

Lecture 3. Topics to be chosen among the following. Exotic solutions to the Serrin problem : minimal free boundary immersions. Overdetermined problem for the Steklov problem. Heat equation and heat kernel. Basic diffusion function and constant heat flow. Constant heat flow and the isoparametric property.