Title: Cheeger and Buser inequalities for magnetic Laplacians on graphs and manifolds

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Abstract: Celebrated classical results in spectral geometry and spectral graph theory are relations between the Cheeger isoperimetric constant and the first nonzero eigenvalue of the Laplace-Operator (Cheeger and Buser inequalities). Recently, there was significant progress relating so-called higher order Cheeger isoperimetric constants to the higher eigenvalues of the Laplacian by Lee/Oveis Gharan/Trevisan 2012 and Funano 2013. In graph theory, these results are related to the very timely topics of expander graphs and spectral clustering.

On the other hand, the classical Laplacian on both graphs and manifolds has generalizations to magnetic Laplacians via the introduction of a magnetic potential in the case of a Riemannian manifold and of a signature on the oriented edges in the case of a graph. It is natural to ask for generalizations of the above mentioned Cheeger and Buser inequalities to magnetic Laplacians.

In this minicourse, we will first introduce magnetic Laplacians on graphs and manifolds and associated (higher order) magnetic Cheeger constants, which are a combination of the frustration index involving the signature or magnetic potential and the geometric expansion rate. We will also discuss fundamental notions like Gauge transformations, switching functions and balanced signatures. We will then discuss the associated (higher order) magnetic Cheeger inequalities, that is, lower bounds of the eigenvalues by the corresponding Cheeger constants and maybe, if time permits, applications to spectral clustering in the graph theoretical setting.

Buser's inequality states that, in the case of a lower bound on the Ricci curvature of a compact Riemannian manifold, there is also an upper bound of the first nonzero Laplace eigenvalue in terms of Cheeger's constant. If we want to transfer this result into the graph theoretical setting, we encounter the challenging problem to define a suitable Ricci curvature notion on graphs. It turns out that a suitable curvature notion is given by the curvature-dimension inequality $CD(K, \infty)$, based on Bochner's formula and Bakry's Γ-calculus. Using this notion, Klartag/Kozma/Ralli/Tetali 2015 were able to transfer and to extend a simple analytic proof of Buser's inequality by Ledoux in 1994 for manifolds into the graph theoretical setting. So, in the next part of the minicourse, I will introduce this analytic curvature notion on graphs.

If time permits, I will also mention a graph theoretical analogue of Bonnet-Myers for this curvature notion. In this middle part of the minicourse there are no magnetic fields or signatures involved.

In the final part of the minicourse, we will return to the magnetic case and introduce, on graphs, a magnetic version of the curvature-dimension inequality $CD(K, \infty)$, involving the signature on oriented edges. We will entend the above-mentioned ideas from the classical non-magnetic case to the magnetic case, and finally discuss higher order magnetic Buser inequalities in the graph case. If time permits, we will also mention, in combination with Lichnerovicz' inequality, a jump phenomenon of the magnetic curvature around a balanced signature.

We like to mention that these results can also be studied in the more general setting of connection Laplacians, but our presentation restricts to the simpler magnetic case to avoid additional technical and notational difficulties.

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