BA2007 PROBLEMS SESSION

YEMON CHOI: Cohomology of commutative, amenable Banach algebras
It is known that, if A is an amenable commutative Banach algebra and X is a symmetric Banach A-bimodule, then

$$H^1(A, X) = H^2(A, X) = 0.$$

Furthermore, if the underlying Banach space of A is ℓ^1, then the same holds for all $H^n, n \in \mathbb{N}$.

Problem: What happens for $n \geq 3$ if we drop the Banach-space assumption on A? In particular, if $A = C[0, 1]$ or $L^1(\mathbb{R})$, does there exist a symmetric bimodule X for which $H^3(A, X) \neq 0$?

GARTH DALES: Hyper-Stonian spaces
Let Ω be a compact space. Then Ω is *hyper-Stonian* if $C(\Omega)$ is a Von Neumann algebra, i.e., $C(\Omega) = F'$ for a Banach space F.

Problem: Suppose that $C(\Omega) = F''$ for a Banach space F. What exactly can we say about Ω? Is it true that there exists a locally compact space X such that $C(\Omega) = C_0(X)''$?

In general, $C(\Omega) = F'' \not\Rightarrow F = C_0(X)$ for some locally compact X. However, this implication *is* true if, in addition, F is a Banach lattice.

COLIN GRAHAM: Riemann localization and summation methods
Let $f \in L^1(\mathbb{T})$. Classical Riemann localization says that $x \notin \operatorname{supp} f$ iff $\sum_{-N}^{N} e^{inx} \widehat{f}(n) \to 0$ pointwise in a neighbourhood of x. The implication '$\Rightarrow$' was discussed for other summation methods in my talk.

Problem: For which summation methods does '$\Leftarrow$' hold?

MATTHEW HEATH: Derivations from the disk algebra into its dual
Let A be the disc algebra. It is known that, if $T : A \to A^*$ is a bounded linear map, then there exists a bounded linear map $\widetilde{T}$ and a measure μ_T on $\mathbb{T}$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "i"'] & A^* \\ C(\mathbb{T}) \arrow[r, "i"] & L^2(\mathbb{T}, \mu_T) \arrow[u, "\widetilde{T}"'] \end{tikzcd}$$

Thus T is weakly compact. Now let $D : A \to A^*$ be a bounded derivation.

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Problem 1. (Suggested by Yemon Choi) Is there a 'nice' choice of μ_D?

Problem 2. Must D be (norm-)compact?

For $\lambda \in \mathbb{D}$, let $D_\lambda(f)(g) := D(\lambda^{-1} f)(\lambda^{-1} g)$. I have shown that D can be approximated in norm by finite-rank derivations iff $D_\lambda \to D$ as $\lambda \to 1$.

Problem 3: Does $D_\lambda \to D$ as $\lambda \to 1$?

ALEXANDER HELEMSKII: Semi-Ruan extreme flatness/projectivity

Let L be a Hilbert space. We say that a left normed $\mathcal{B}(L)$-module X is *semi-Ruan extremely flat* (resp. *extremely flat*) if, for every isometric morphism $\alpha : Y \to Z$ of semi-Ruan modules (resp. arbitrary normed modules), the operator $\alpha \otimes_{\mathcal{B}} 1_X : Y \otimes_{\mathcal{B}} Z \to Z \otimes_{\mathcal{B}} X$ is also isometric. It is known that, for an arbitrary Hilbert space H, the left $\mathcal{B}(L)$-module $L \dot{\otimes} H$ with $a \cdot \xi := (a \dot{\otimes} 1_H)(\xi)$ is semi-Ruan extremely flat. In this connection, two natural questions arise.

Problem 1: Are the modules $L \dot{\otimes} H$ extremely flat?

It is known only that these modules have no stronger property of extreme projectivity (see below). They are not even projective in the usual sense of normed homology. This was proved in 1994, and the existing proof is rather difficult.

We say that a normed left $\mathcal{B}(L)$-module P is *semi-Ruan extremely projective* (resp. *extremely projective*) if, for every coisometric morphism $\tau : Z \to Y$ of semi-Ruan modules (resp. arbitrary normed modules) and every bounded morphism $\phi : P \to Y$, there exists a lifting morphism $\psi : P \to Z$ of ϕ such that $\|\psi\| = \|\phi\|$. It is easy to see that every semi-Ruan extremely projective (resp. extremely projective) module is extremely flat in the relevant sense.

Problem 2: Are the modules $L \dot{\otimes} H$ semi-Ruan extremely projective?

From what was said in Problem 1, it is clear that they are not just extremely projective.

KRZYSZTOF JAROSZ: Some problems on uniform algebras

Problem 1: Assume $A \subseteq C(X)$ is a function algebra with $X = \mathfrak{M}(A) =$ the maximal ideal space of A, $M \subset A$, $\operatorname{codim}(M) = n < \infty$. Is the following implication true:

$$(\forall f \in M \exists x \in X \qquad f(x) = 0) \Longrightarrow (\exists x \in X \forall f \in M \qquad f(x) = 0)?$$

The implication is true (but not at all obvious) for $A = C(X)$ and for any n; it is also true for $n = 1$ and any Banach algebra (classical Gleason-Kahane-Żelazko Theorem), there is no known counterexample, it is open even for $A =$ the disc algebra.
For more background start with: `http://www.siue.edu/MATH/kj_papers/when.pdf`

Problem 2: Assume A is a Banach algebra, G is a linear multiplicative functional on A, and Δ is a linear functional with a very small norm. Then $F = G + \Delta$ is obviously almost multiplicative:

$$|F(fg) - F(f)F(g)| \leq \varepsilon \|f\| \|g\|.$$

Is this the only way to construct an almost multiplicative functional? Is any almost multiplicative functional near a multiplicative one?

The answer is YES for $A = C(X)$ (easy), also YES for the disc algebra and some related algebras (not trivial). In general the answer is NO, there are even uniform algebras with almost multiplicative functionals far from any multiplicative one. The problem is open for a number of important Banach algebras including $H^\infty(\mathbb{D})$.
For more background see: `http://www.siue.edu/MATH/kj_papers/almost_mult.pdf`

Problem 3: In general there is little relation between the geometry of a Banach space A and Banach algebra multiplications we may define on that space. For example l^1 can be made into a Banach algebra with the pointwise multiplication or with the convolution multiplication. On the other hand it has been known for almost 50 years that two isometric uniform algebras must automatically be identical as algebras. To what extent is this stable? Assume A is a uniform algebra, B is Banach algebra and the Banach-Mazur distance between these Banach spaces is small: $d_{B-M}(A, B) < 1 + \varepsilon$. Must B be automatically identical with A as an algebra, or at least must it share with A some crucial properties?

There are some very partial answers but in general the question is open. For more background information start with: *K. Jarosz, Perturbations of Banach Algebras, Springer-Verlag, Lecture Notes in Math. 1120, 1985.*

Problem 4: Is there a uniform algebra $A \subset C(\mathbb{D})$, with $\mathbb{D}$ = unit disc = the maximal ideal space of A, and such that the Shilov boundary of A is contained in the interior of $\mathbb{D}$?

NIELS JAKOB LAUSTSEN: Banach spaces supporting few operators
These problems are concerned with the Banach algebra $\mathcal{B}(E)$ of all bounded linear operators on a Banach space E and the ideal $\mathcal{K}(E)$ of compact operators. They are motivated by the famous 'scalar plus compact' problem (whether there exists an infinite-dimensional Banach space E such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C}I$).

Problem 1: Does there exist an infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is separable?

Problem 2: Does there exist an infinite-dimensional Banach space E, not isomorphic to c_0 or ℓ_p for any $p \in [1, \infty)$, such that $\mathcal{K}(E)$ is a maximal ideal in $\mathcal{B}(E)$?

Another problem, raised during my conference talk, is the following:

Problem 3: Classify the closed ideals in $\mathcal{B}(G)$ and $\mathcal{B}(G')$, where

$$G = (\ell_1^1 \oplus \ell_1^2 \oplus \ell_1^3 \oplus \cdots \oplus \ell_1^n \oplus \dots)_{\ell_2},$$

and G' is its dual, namely

$$G' = (\ell_\infty^1 \oplus \ell_\infty^2 \oplus \ell_\infty^3 \oplus \cdots \oplus \ell_\infty^n \oplus \dots)_{\ell_2}.$$

RICK LOY: Approximate amenability and bounded approximate identities
(with Fereidoun Ghahramani and Yong Zhang)
The Banach algebra A is *approximately amenable* if every continuous derivation $D : A \to X^*$ is approximately inner, i.e. D is the strong limit of $\operatorname{ad} x_i^*$ for some net $(x_i^*) \subset X^*$.

Problem 1: Does A and B approximately amenable imply $A \oplus B$ approximately amenable? This is unknown even if $A = B$.

Problem 2: Does an approximately amenable Banach algebra have a *two-sided* bounded approximate identity?

Note:

- If A has left and right bounded approximate identities, then it has a two-sided one.
- If A is a Banach sequence algebra with bounded approximate identity containing c_{00} as a dense subalgebra, then A is approximately amenable.
- If $A \oplus A$ is approximately amenable, then A has a two-sided bounded approximate identity.
- If A, B are approximately amenable and one of them has a bounded approximate identity, then $A \oplus B$ is approximately amenable.
- If $\overline{AA^*}$ is complemented in A^* and A is boundedly approximately amenable, then A has a two-sided approximate identity.

MATTHIAS NEUFANG: Extreme amenability
A topological group G is *extremely amenable* if there exists a left-invariant mean on $\mathrm{LUC}(G)$ that is multiplicative.

Problem: Is there a Banach-algebra concept that captures extreme amenability (in the spirit of Johnson's well-known characterization of amenability)?

VOLKER RUNDE: Amenability of Figà-Talamanca–Herz algebras
Let G be a locally compact group, let $p \in (1, \infty)$, and let $A_p(G)$ denote the corresponding Figà-Talamanca–Herz algebras ($A_2(G) = A(G)$, Eymard's Fourier algebra).

Problem: For which G is $A_p(G)$ amenable?

Conjecture: $A_p(G)$ is amenable for some $p \in [1, \infty)$ iff G is almost abelian (i.e. has an abelian subgroup of finite index).
The following facts are known:

- G almost abelian implies that $A_p(G)$ is amenable for all $p \in (1, \infty)$.
- $A(G)$ amenable implies that G is almost abelian (Forrest–Runde, 2005)
- $A_p(G)$ 1-amenable for some $p \in (1, \infty)$ implies that G is abelian (Runde, 2007)
- $A_p(G)$ is operator amenable for *some* $p \in (1, \infty)$ iff $A_p(G)$ is operator amenable for *all* $p \in (1, \infty)$, and these occur iff G is amenable.

NICO SPRONK: Segal algebras
Let G be a locally compact group.

Problem: Is it true that, if $S_1(G), S_2(G)$ are two Segal algebras in $L^1(G)$ (in the classical sense of Reiter), then $S_1(G) \cap S_2(G) \neq \emptyset$?

This is known to be true for abelian G, since each Segal algebra contains $\mathcal{F}^{-1}(\mathcal{F}(L^1(G)) \cap C_c(G))$, where $\mathcal{F}$ is the Fourier transform. It is also true for compact G, since each Segal algebra contains $\mathcal{T}(G)$, the 'trigonometric functions'. Is it true for non-unimodular groups, such as the $ax + b$ group?
(The motivation is to characterize the Feichtinger algebra as the minimal Segal algebra $S(G)$ in $L^1(G)$ such that $A(G) \cdot S(G) \subset S(G)''$.)

THOMAS TONEV: Non-linear maps between uniform algebras
Denote by $\sigma(T)$ the spectrum of T, and by $\sigma_\pi(T)$ the peripheral spectrum.

Problem 1. Let A and B be uniform algebras and $T : A \to B$ be a (perhaps non-linear) surjective mapping such that

$$(*) \qquad \sigma_\pi((Tf)(Tg)) \subset \sigma(fg),$$

Does it follow that T is an algebra homomorphism?

Problem 2. The same as in Problem 1, with $(*)$ replaced by

$$(**) \qquad \sigma_\pi(\lambda Tf + \mu Tg) \subset \sigma(\lambda f + \mu g) \quad (\lambda, \mu \in \mathbb{C}).$$

In the case when there are equalities in $(*)$ and $(**)$, T is an algebra isomorphism (Luttman–Tonev for $(*)$, Rao–Tonev–Toneva for $(**)$). If $B = \mathbb{C}$ and if T is in addition linear, then Problem 2 is simply Gleason–Kahane–Żelazko's theorem for uniform algebras.

JAROSLAV ZEMÁNEK: Spectral theory of Hilbert-space operators

Problem 1: Let A be an operator on a Hilbert space. Denote by $W(A)$ its numerical range. Suppose that $1 \in \partial W(A^k)$ for all $k \geq 1$. What can be concluded about $W(A)$? Must it be included in the unit disk, or even in $[-1, 1]$? Does it follow that $1 \in \sigma(A)$?

Problem 2: For normal operators A, B, it is known that

$$\operatorname{dist}(\sigma(A), \sigma(B)) \leq \|A - B\|.$$

Does there exist a complex-valued function ϕ defined on normal operators such that, for all normal A and B

$$\phi(A) \in \sigma(A) \qquad \text{and} \qquad |\phi(A) - \phi(B)| \leq \|A - B\|?$$

Problem 3: (with Mostafa Mbekhta) Let $A(z)$ be an analytic family of compact operators on the open unit disk $\mathbb{D}$. Suppose that $A(z_n)$ are quasi-nilpotent for some infinite sequence $z_n \to 0$. Does it follow that the operators $A(z)$ are quasi-nilpotent for all $z \in \mathbb{D}$?

Problem 4: Let $A(z)$ be an analytic family of bounded operators on $\mathbb{D}$ such that the spectral radius of $A(z)$ equals 1 for all $z \in \mathbb{D}$. It is known that the peripheral spectrum of $A(z)$ is then independent of z. What other properties remain constant? For example, if the resolvent of $A(z)$ has a pole at 1 for some $z \in \mathbb{D}$, must the same be true for all $z \in \mathbb{D}$?