Rapidly and slowly oscillating periodic solutions of a delayed van der Pol oscillator

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Abstract

In this paper, we introduce a method to prove existence of several rapidly and slowly oscillating periodic solutions of a delayed van der Pol oscillator. The proof is a combination of pen and paper analytic estimates, the contraction mapping theorem and a computer program using interval arithmetic. Using this approach we extend some existence results obtained by Nussbaum in [Ann. Mat. Pura Appl., **4** (101), 263–306, 1974].

Keywords

Delay differential equations · Rigorous numerics · Rapid and slow periodic solutions · Contraction mapping theorem · van der Pol equation

1 Introduction

In 1920, to model oscillations of some electric circuits, van der Pol proposed one of the most influential system of nonlinear differential equations, see [1]. Since then, variants of the so-called van der Pol oscillator have been proposed as mathematical models of various real-world processes exhibiting limit cycles when the rate of change of the state variables depend only on their current states. However, there are many processes where this relation is also influenced by past values of the system in question. To model these processes, one may want to consider the use of functional differential equations, see [2, 3, 4, 5].

In [7], Grafton establishes existence of periodic solutions to

$$\ddot{y}(t) - \varepsilon\dot{y}(t)(1 - y^2(t)) + y(t - \tau) = 0, \quad \varepsilon, \tau > 0, \tag{1}$$

a van der Pol equation with a retarded position variable. His results are based on his periodicity results developed in [8]. In [9], using slightly different notations, Nussbaum considers the more general class of equations

$$\ddot{y}(t) - \varepsilon\dot{y}(t)(1 - y^2(t)) + y(t - \tau) - \kappa y(t) = 0, \quad \varepsilon, \tau > 0, \kappa \in \mathbb{R}, \tag{2}$$

and establishes the following result.

Theorem 1.1 (Nussbaum, 1974). *If $\kappa < 0$, $-\kappa\tau < \varepsilon$ and $-\frac{1}{2}\kappa\tau^2 \leq 1$, then equation (2) has a nonzero periodic solution of period greater than 2τ.*

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We refer to (2) as Nussbaum's equation. The techniques that Nussbaum uses are sophisticated fixed point arguments, however, as he remarks, he has

> *to restrict the size of $|\kappa|$ in order to guarantee that the zeros of y are at least a distance τ apart*, p. 287 of [9].

Moreover, he mentions that numerical simulations suggest the existence of periodic solutions to (2) for a large range of $\kappa < 0$. Thus, it seems natural to try to develop a computational tool in order to establish existence results for periodic solutions to (2) for parameter values seemingly inaccessible using Nussbaum's methods. Before formulating our results, we need the following definition.

Definition 1.2. Let $y(t)$ be a periodic solution to (2) with the minimal period p. Assume that $0 = z_1 < z_2 < \ldots < z_n = p$ are such that $y(z_i) = 0,\ i = 1, \ldots, n$, and that these are the only zeroes of y. If $z_{i+1} - z_i > \tau$ for all $i = 1, \ldots, n-1$ then y is a *slowly oscillating periodic solution.* Similarly, y is said to be a *rapidly oscillating periodic solution* if $z_{i+1} - z_i < \tau$ for all $i = 1, \ldots, n-1$.

Our aim is to prove the following statements, whose proofs are a combination of pen and paper analytic estimates and computer programming using interval arithmetic.

Theorem 1.3. *Let $\varepsilon = 0.15$ and $\tau = 2$. For 277 distinct values of $\kappa \in [-3.8521, 0.1379]$, there exists a nontrivial periodic solution of (2). As for the range of periods p of these solutions we have that $p \in [3.69, 11.24]$.*

The parameter values of Theorem 1.3 were obtained by performing a numerical continuation on κ with step size 0.015.

Theorem 1.4. *Let $\varepsilon = 0.25$ and $\tau = 5$. For 383 distinct parameter values of $\kappa \in [-4.6814, -0.4762]$, there exists a nontrivial periodic solution of (2). As for the range of the periods p of these solutions we have that $p \in [3.15, 5.26]$.*

The parameter values of Theorem 1.4 were also obtained by performing a numerical continuation on κ with varying step size.

Remark 1.5. The norms of the computed solutions are shown in Figure 1 while their periods are plotted in Figure 2. Also, some illustrative time profiles can be seen in Figure 3 and 4; these figures indicate that, indeed, there are both slowly and rapidly oscillating solutions to (2). Furthermore, our results relax the periodicity condition in Nussbaum's theorem. The trade-off here clearly is that our method applies to a given specific equation. Also, unlike in the case of finite dimensional dynamical systems, see [6], we do not have tools to address stability properties of these periodic solutions. Nevertheless, since these periodic solutions are visible via numerical simulations suggests their stability.

The proofs of Theorem 1.3 and Theorem 1.4 are based on adapting ideas from [10] and [11] to the context of second order delay equations with cubic nonlinearities. Moreover, we would like to mention that Nussbaum considers in [9] a large family of functional differential equations of which (2) is a particular member. Establishing existence results of periodic solutions to other members of the family of equations considered in [9] is an ongoing project.

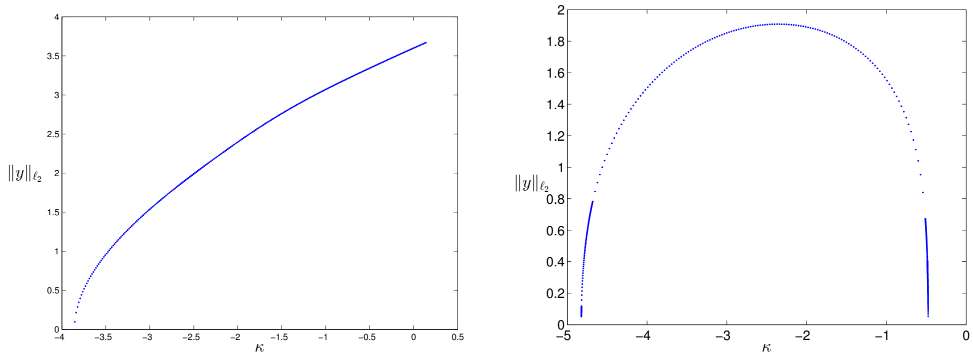

Figure 1: **Left:** The norm of the computed 267 solutions of Theorem 1.3 with respect to the parameter κ for $\tau = 2$ and $\varepsilon = 0.15$. **Right:** The norm of the computed 383 solutions of Theorem 1.4 with respect to the parameter κ for $\tau = 5$ and $\varepsilon = 0.25$.

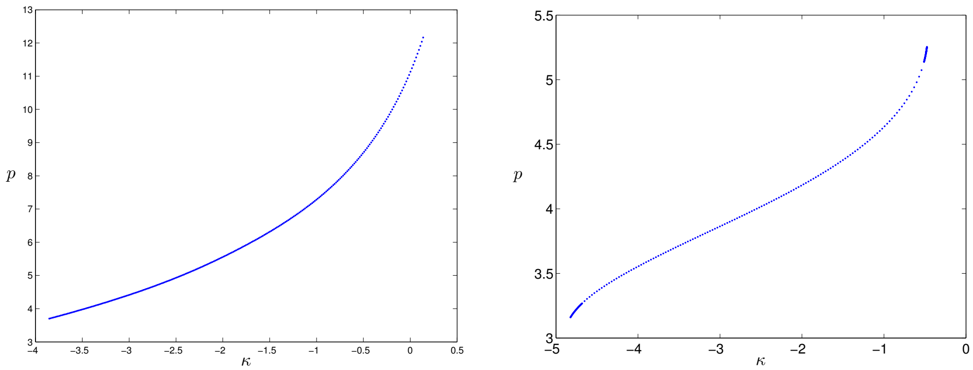

Figure 2: **Left:** The period of the computed 267 solutions of Theorem 1.3 with respect to the parameter κ for $\tau = 2$ and $\varepsilon = 0.15$. **Right:** The period of the computed 383 solutions of Theorem 1.4 with respect to the parameter κ for $\tau = 5$ and $\varepsilon = 0.25$.

2 Setting up a fixed point problem

If $y(t)$ is a periodic solution of (2) with a period $p > 0$, then

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikLt}, \tag{3}$$

where $L = \frac{2\pi}{p}$ and the c_k are complex numbers satisfying $c_{-k} = \overline{c_k}$, since $y \in \mathbb{R}$. Denoting the real and the imaginary part of c_k respectively by a_k and b_k, one gets that $a_k = a_{-k}$ and $b_k = -b_{-k}$. As a result, $b_0 = Im(c_0) = 0$ and so it is not a variable. An equivalent expansion for (3) is given by

$$y(t) = a_0 + 2\sum_{k=1}^{\infty} \left[a_k \cos kLt - b_k \sin kLt\right]. \tag{4}$$

As the frequency L of (3) is not known *a-priori*, it is left as a variable. Since periodic solutions of analytic delay differential equations are analytic (e.g. see [12]), their Fourier coefficients decay faster then any algebraic decay. This will motivate the

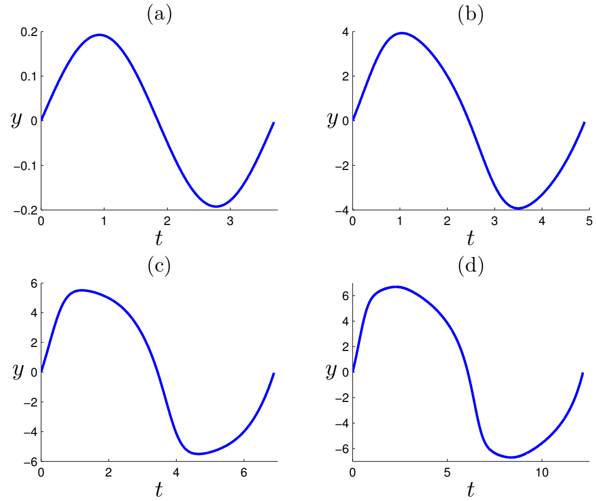

Figure 3: Some of the computed 267 solutions with $\tau = 2$ and $\varepsilon = 0.15$. On panel (a), (b), (c) and (d), $\kappa = -3.8521,\ -2.5171,\ -1.1821$ and 0.1379, respectively.

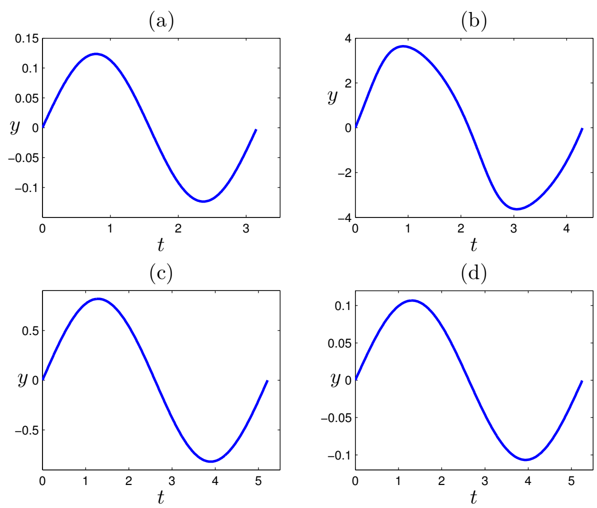

Figure 4: Some of the computed 383 solutions with $\tau = 5$ and $\varepsilon = 0.25$. On panel (a), (b), (c) and (d), $\kappa = -4.8232,\ -1.6214,\ -0.4861$ and -0.4761, respectively.

choice of Banach space in which we will embed the Fourier coefficients. Let

$$x_k \stackrel{\text{def}}{=} \begin{cases} (L, a_0), & k = 0 \\ (a_k, b_k), & k > 0 \end{cases} \tag{5}$$

and $x \stackrel{\text{def}}{=} (x_0, x_1, \ldots, x_k, \ldots)$, and denote the first and the second component of x_k by $x_{k,1}$ and $x_{k,2}$, respectively. Given a growth rate $s > 0$, consider the weight functions

$$\omega_k^s = \begin{cases} 1, & k = 0 \\ |k|^s, & k \neq 0 \end{cases} \tag{6}$$

which are used to define the norm

$$\|x\|_s \stackrel{\text{def}}{=} \sup_{k\geq 0} |x_k|_\infty \omega_k^s, \tag{7}$$

where $|x_k|_\infty = \max\{|x_{k,1}|, |x_{k,2}|\}$.

Lemma 2.1. *For a given $s > 1$, consider the space of sequences with algebraically decaying tails*

$$\Omega^s \stackrel{\text{def}}{=} \{x = (x_0, x_1, x_2 \ldots) : \|x\|_s < \infty\}$$

is a Banach space. Moreover, assume that $y(t)$ given by (4) is a periodic solution of Nussbaum equation (2) and consider the associated x given by (5). Then for any fixed $s > 1$, the space Ω^s contains x.

Proof. The fact that Ω^s is a Banach space is standard and the proof is omitted. Let $y(t)$ given by (4) a periodic solution of (2) and let x the associated vector given component-wise by (5). Since $y(t)$ is a periodic solution of the analytic delay differential equation (2), it follows from [12] that it is analytic. Therefore the Fourier coefficients of $y(t)$ decay exponentially fast to zero. Hence, the sequences $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ converge to zero faster than any algebraic decay. This implies that for any $s > 1$, $x \in \Omega^s$. □

Formally, using (3), we get

$$\dot{y}(t) = \sum_{k=-\infty}^{\infty} c_k ikL e^{ikLt},$$

$$\ddot{y}(t) = \sum_{k=-\infty}^{\infty} -c_k k^2 L^2 e^{ikLt}$$

and

$$y(t-\tau) = \sum_{k=-\infty}^{\infty} c_k e^{-ikL\tau} e^{ikLt}.$$

Thus (2) becomes

$$\sum_{k=-\infty}^{\infty} \left[-k^2L^2 - \varepsilon ikL - \kappa + e^{-ikL\tau}\right] c_k e^{ikLt} + \varepsilon \sum_{k_1=-\infty}^{\infty} c_{k_1} e^{ik_1Lt} \sum_{k_2=-\infty}^{\infty} c_{k_2} e^{ik_2Lt} \sum_{k_3=-\infty}^{\infty} c_{k_3} ik_3 L e^{ik_3Lt} = 0. \tag{8}$$

To obtain the Fourier coefficients in (3), one takes the inner product on both sides of (8) with e^{ikLt}, $k \in \mathbb{Z}$, yielding

$$g_k \stackrel{\text{def}}{=} \left[-k^2L^2 - \varepsilon ikL - \kappa + e^{-ikL\tau}\right] c_k + i\varepsilon L \sum_{k_1+k_2+k_3=k} c_{k_1} c_{k_2} c_{k_3} k_3 = 0.$$

Observing that

$$
\begin{aligned}
S_k &\stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} c_{k_1}c_{k_2}c_{k_3}k_3 \\
&= \sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} c_{k_1}c_{k_2}c_{k_3}(k-k_1-k_2) \\
&= k\sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} c_{k_1}c_{k_2}c_{k_3} - 2S_k,
\end{aligned}
$$

we get that

$$
g_k = \left[-k^2L^2 - \varepsilon ikL - \kappa + e^{-ikL\tau}\right]c_k + \frac{i\varepsilon kL}{3}\sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} c_{k_1}c_{k_2}c_{k_3} = 0,
$$

which is a countable system of nonlinear complex-valued equations. Using (5), one gets that

$$
\begin{aligned}
Re(g_k)(x) =&(-k^2L^2 - \kappa + \cos kL\tau)a_k + (\varepsilon kL + \sin kL\tau)b_k \\
&+ \frac{\varepsilon kL}{3}\sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} -a_{k_1}a_{k_2}b_{k_3} - 2a_{k_1}b_{k_2}a_{k_3} + b_{k_1}b_{k_2}b_{k_3} \\
Im(g_k)(x) =&(-k^2L^2 - \kappa + \cos kL\tau)b_k - (\varepsilon kL + \sin kL\tau)a_k \\
&+ \frac{\varepsilon kL}{3}\sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} a_{k_1}a_{k_2}a_{k_3} - 2a_{k_1}b_{k_2}b_{k_3} - b_{k_1}b_{k_2}a_{k_3}.
\end{aligned}
$$

For a periodic function $y(t)$ with expansion (3), let us define the map

$$
h(x) = y(0) = a_0 + \sum_{k=1}^{\infty} a_k.
$$

Hence, the map f whose zeros correspond to periodic solutions (oscillating around 0) of (2) is given component-wise by

$$
f_k(x) \stackrel{\text{def}}{=} \begin{cases} \begin{pmatrix} h(x) \\ Re(g_0)(x) \end{pmatrix}, & k=0; \\ \begin{pmatrix} Re(g_k)(x) \\ Im(g_k)(x) \end{pmatrix}, & k>0. \end{cases} \tag{9}
$$

As in the case of x, denote the first and the second component of f_k by $f_{k,1}$ and $f_{k,2}$, respectively. For the sake of simplicity of the presentation, let us introduce

$$
R_k(L) \stackrel{\text{def}}{=} \begin{pmatrix} -k^2L^2 - \kappa + \cos kL\tau & \varepsilon kL + \sin kL\tau \\ -(\varepsilon kL + \sin kL\tau) & -k^2L^2 - \kappa + \cos kL\tau \end{pmatrix}. \tag{10}
$$

Also, for given three bi-infinite vectors $a = (a_k)_{k\in\mathbb{Z}}$, $b = (b_k)_{k\in\mathbb{Z}}$ and $c = (c_k)_{k\in\mathbb{Z}}$, we define the discrete cubic convolution term component-wise by

$$
(a*b*c)_k \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=k \\ k_j\in\mathbb{Z}}} a_{k_1}b_{k_2}c_{k_3}, \quad k\in\mathbb{Z} \tag{11}
$$

to write

$$f_k(x) = R_k(L) \begin{pmatrix} a_k \\ b_k \end{pmatrix} + \frac{\varepsilon k L}{3} \begin{pmatrix} -(a*a*b)_k - 2(a*b*a)_k + (b*b*b)_k \\ (a*a*a)_k - 2(a*b*b)_k - (b*b*a)_k \end{pmatrix} \quad (12)$$

for $k \geq 1$. In setting up our fixed point problem, the following is crucial.

Lemma 2.2. *The operator $f = \{f_k\}_{k\geq 0}$, defined component-wise in (9), has the property that $f : \Omega^s \to \Omega^{s-2}$.*

Proof. It is not difficult to see that

$$\sup_{k\geq 0} \left| \left(-k^2L^2 - \kappa + \cos kL\tau\right) a_k + (\varepsilon k L + \sin kL\tau)\, b_k \right| \omega_k^{s-2} < \infty$$

and

$$\sup_{k\geq 0} \left| (-\varepsilon k L - \sin kL\tau)\, a_k + \left(-k^2L^2 - \kappa + \cos kL\tau\right) b_k \right| \omega_k^{s-2} < \infty$$

for $x \in \Omega^s$.

To proceed, for vectors $u, v \in \mathbb{R}^{m\times n}$, we introduce $u \ll v$ to denote the fact that $u_{i,j} \leq v_{i,j},\ i = 1, \ldots, m,\ j = 1, \ldots, n$. Since $x \in \Omega^s$, by definition,

$$|a_k| \leq \frac{\|x\|_s}{\omega_k^s} \quad \text{and} \quad |b_k| \leq \frac{\|x\|_s}{\omega_k^s}, k \in \mathbb{Z}.$$

Now, from Lemma 2.1 in [13], there are $C_{k,1}$ and $C_{k,2}$ positive constants uniformly bounded so that

$$\left| \begin{pmatrix} -(a*a*b)_k - 2(a*b*a)_k + (b*b*b)_k \\ (a*a*a)_k - 2(a*b*b)_k - (b*b*a)_k \end{pmatrix} \right| \ll \frac{1}{\omega_k^s} \begin{pmatrix} C_{k,1} \\ C_{k,2} \end{pmatrix},\ k \geq 0.$$

This implies that

$$\|f(x)\|_{s-2} = \sup_{k\geq 0} |f_k|_\infty \omega_k^{s-2} < \infty,$$

that is, $f(x) \in \Omega^{s-2}$. □

Remark 2.3. *We want to emphasize that the idea of deriving a set of analytic estimates to bound the truncation error term are originally used in [13, 15, 16] to develop constructive proofs of existence of equilibria to partial differential equations. However, in the present paper, we decided to write and prove all estimates explicitly to keep this work self-contained.*

The following result establishes the correspondence between the periodic solutions of (2) and the zeros of f. Its proof is omitted and a similar statement with a proof can be found for example in [10].

Lemma 2.4. *For a sequence $x = (x_0, x_1, x_2, \ldots) \in \Omega^s$ given in (5), $f(x) = 0$ if and only if $y(t)$ in (4) is a solution to (2) such that $y(0) = a_0 + 2\sum_{k=1}^{\infty} a_k = 0$.*

Now, we rewrite $f(x) = 0$ as a fixed-point equation $T(x) = x$ in Ω^s where T is a Newton-like operator. In order to define T, first consider $f^{(m)} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, a finite dimensional projection f, whose k-th component is given by

$$f_k^{(m)}(x_0, \ldots, x_{m-1}) \stackrel{\text{def}}{=} f_k(x_0, \ldots, x_{m-1}, 0_\infty),\ k = 0, \ldots, m-1,$$

where $0_\infty = (0)_{j\geq 0}$. Hereafter, we identify $\bar{x} = (\bar{L}, \bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_{m-1}, \bar{b}_{m-1})$ with $(\bar{x}, 0_\infty)$ if $\bar{x} \in \mathbb{R}^{2m}$ such that $f^{(m)}(\bar{x}) \approx 0$. Now we define the finite part of T using numerics while its tail will be defined analytically. To define the finite part, consider the later described computational parameter $M = 3m - 2$ (e.g. see Lemma 3.1), and let $A_M \in \mathbb{R}^{2M\times 2M}$ be a numerical approximation of the inverse of $D_x f^{(M)}(\bar{x})$. Assume that

$$\|I_M - A_M D_x f^{(M)}(\bar{x})\|_\infty < 1, \tag{13}$$

which implies that A_M is an invertible matrix.

Furthermore, let

$$\Lambda_k \stackrel{\text{def}}{=} \frac{\partial f_k}{\partial x_k}(\bar{x}) = \begin{pmatrix} \varrho_k & \delta_k \\ -\delta_k & \varrho_k \end{pmatrix},$$

where

$$\varrho_k \stackrel{\text{def}}{=} -k^2 \bar{L}^2 - \kappa + \cos k\bar{L}\tau$$

and

$$\delta_k \stackrel{\text{def}}{=} \varepsilon k\bar{L} + \sin k\bar{L}\tau - 2\varepsilon k\bar{L}\left(\frac{\bar{a}_0^2}{2} + \sum_{k_1=1}^{m-1}\left(\bar{a}_{k_1}^2 + \bar{b}_{k_1}^2\right)\right).$$

Since $\kappa < 1$, a sufficient condition for the invertibility of Λ_k for all $k > M$ is

$$m > \frac{1}{3}\left(\frac{\sqrt{1-\kappa}}{\bar{L}} + 2\right). \tag{14}$$

Indeed, (14) and $k \geq M$ imply $\varrho_k < 0$ and thus $\varrho_k^2 + \delta_k^2 > 0$ for all $k \geq M$. Define the linear operator $A : \Omega^s \to \Omega^{s+2}$ by

$$Ax \stackrel{\text{def}}{=} \begin{cases} (A_M x_F)_k, & k = 0, \ldots, M-1 \\ \Lambda_k^{-1} x_k, & k \geq M. \end{cases} \tag{15}$$

In this definition,

$$\Lambda_k^{-1} = \frac{1}{\varrho_k^2 + \delta_k^2}\begin{pmatrix} \varrho_k & -\delta_k \\ \delta_k & \varrho_k \end{pmatrix},$$

and

$$x_F = (x_0, x_1, \ldots, x_{M-1}) \in \mathbb{R}^{2M}$$

is the finite dimensional projection of $x \in \Omega^s$. The conditions (13) and (14) imply the invertibility of the operator A defined in (15).

Notice that m in (14) is an explicit lower bound on the dimension of the finite dimensional projection for A to be an invertible linear operator. Now, we complete the verification of the definition of A, that is, we show that $A : \Omega^s \to \Omega^{s+2}$. Given a matrix M denote by $|M|$ the matrix whose components are the component-wise absolute values of M.

Lemma 2.5. *Let $\bar{L} > 0$ and $\bar{a}_0 \in \mathbb{R}$, and define*

$$\rho \stackrel{\text{def}}{=} \frac{M^2}{M^2\bar{L}^2 - (1-\kappa)} > 0$$

and

$$\Xi \stackrel{\text{def}}{=} \begin{pmatrix} \rho & \rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M} + \frac{1}{M^2}\right) \\ \rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M} + \frac{1}{M^2}\right) & \rho \end{pmatrix},$$

where

$$S \stackrel{\text{def}}{=} \frac{\bar{a}_0^2}{2} + \sum_{k_1=1}^{m-1} \left(\bar{a}_{k_1}^2 + \bar{b}_{k_1}^2\right).$$

Then for all $k \geq M$,

$$|\Lambda_k^{-1}| \leq_{cw} \frac{1}{k^2}\Xi.$$

Thus $A : \Omega^s \to \Omega^{s+2}$.

Proof. From (14),

$$|\varrho_k| = -\cos k\bar{L}\tau + k^2\bar{L}^2 + \kappa \geq -1 + k^2\bar{L}^2 + \kappa \geq k^2\left(\bar{L}^2 - \frac{1-\kappa}{M^2}\right) = \frac{k^2}{\rho}.$$

Thus

$$\frac{1}{|\varrho_k|} \leq \rho\frac{1}{k^2},$$

and then

$$\left|\frac{\varrho_k}{\varrho_k^2 + \delta_k^2}\right| \leq \frac{|\varrho_k|}{\varrho_k^2} = \frac{1}{|\varrho_k|} \leq \rho\frac{1}{k^2}.$$

Finally, since $\varepsilon > 0$, $|\delta_k| \leq \varepsilon k\bar{L}|1-2S| + 1$, we get that

$$\begin{aligned}\left|\frac{\delta_k}{\varrho_k^2 + \delta_k^2}\right| &\leq \frac{\varepsilon k\bar{L}|1-2S|+1}{\varrho_k^2 + \delta_k^2} \\ &\leq \frac{\varepsilon k\bar{L}|1-2S|+1}{\varrho_k^2} \\ &\leq \frac{\rho^2(\varepsilon k\bar{L}|1-2S|+1)}{k^4} \\ &\leq \frac{1}{k^2}\rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M} + \frac{1}{M^2}\right).\end{aligned}$$

That is, for $k \geq M$,

$$\left|\Lambda_k^{-1}\right| \ll \frac{1}{k^2}\begin{pmatrix} \rho & \rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M} + \frac{1}{M^2}\right) \\ \rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M} + \frac{1}{M^2}\right) & \rho \end{pmatrix}.$$

Thus, for an $x \in \Omega^s$,

$$\begin{aligned}\|Ax\|_{s+2} &= \max\left\{\max_{k=0,\dots,M-1}\left\{|(A_M x_F)_k|_\infty \omega_k^{s+2}\right\}, \sup_{k\geq M}\left\{|\Lambda_k^{-1}x_k|_\infty \omega_k^{s+2}\right\}\right\} \\ &\leq \max\left\{\max_{k=0,\dots,M-1}\left\{|(A_M x_F)_k|_\infty \omega_k^{s+2}\right\}, \|\Xi\|_\infty \sup_{k\geq M}\left\{|x_k|_\infty \omega_k^s\right\}\right\}. \qquad \square\end{aligned}$$

The conclusion of Lemma 2.5 allows us to define the Newton-like operator $T : \Omega^s \to \Omega^s$ by

$$T(x) = x - Af(x). \tag{16}$$

By injectivity of the operator A, the fixed points of T are in one-to-one correspondence with the zeros of f. Therefore, we now focus on the computation of the fixed points of T. To achieve this new task, we use the radii polynomial approach.

3 The radii polynomial approach

Let

$$B(r) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} \left[-\frac{r}{\omega_k^s}, \frac{r}{\omega_k^s} \right]^2 \tag{17}$$

be the ball of radius $r > 0$ in Ω^s centered at the origin, and

$$B_{\bar{x}}(r) \stackrel{\text{def}}{=} \bar{x} + B(r) \tag{18}$$

be the ball centered at $\bar{x}$. In order to show that T is a contraction, we derive component-wise bounds $Y_k = \begin{pmatrix} Y_{k,1} \\ Y_{k,2} \end{pmatrix}$, $Z_k = \begin{pmatrix} Z_{k,1} \\ Z_{k,2} \end{pmatrix} \in \mathbb{R}^2$ for $k \geq 0$ such that

$$|[T(\bar{x}) - \bar{x}]_k| = |[-Af(\bar{x})]_k| \ll Y_k \tag{19}$$

and

$$\sup_{b,c \in B(r)} |[DT(\bar{x} + b)c]_k| = \sup_{u,v \in B(1)} |[DT(\bar{x} + ru)rv]_k| \ll Z_k(r). \tag{20}$$

Lemma 3.1. *For all $k \geq M = 3m - 2$, $[Af(\bar{x})]_k = (0, 0)^T$.*

Proof. If $k_1, k_2, k_3 \in \mathbb{Z}$ satisfy $k_1 + k_2 + k_3 = k \geq M = 3m - 2$, then there exists $j \in \{1, 2, 3\}$ such that $k_j \geq m$. Since, $\bar{a}_\ell = \bar{b}_\ell = 0$ for all $\ell \geq m$, then

$$\begin{aligned} f_{k,1}(\bar{x}) =& (-k^2\bar{L}^2 - \kappa + \cos k\bar{L}\tau)\bar{a}_k + (\varepsilon k\bar{L} + \sin k\bar{L}\tau)\bar{b}_k \\ &+ \frac{\varepsilon k\bar{L}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_j \in \mathbb{Z}}} -\bar{a}_{k_1}\bar{a}_{k_2}\bar{b}_{k_3} - 2\bar{a}_{k_1}\bar{b}_{k_2}\bar{a}_{k_3} + \bar{b}_{k_1}\bar{b}_{k_2}\bar{b}_{k_3} = 0, \quad \forall\, k \geq M. \end{aligned}$$

A similar argument shows that $f_{k,2}(\bar{x}) = 0$. This implies that $f_k(\bar{x}) = (0, 0)^T$, for all $k \geq M$. Finally, recalling (15), we get that

$$[Af(\bar{x})]_k = \Lambda_k^{-1} f_k(\bar{x}) = (0, 0)^T, \quad \forall\, k \geq M. \qquad \square$$

A consequence of the previous result is that for $k \geq M$, we let $Y_k = (0, 0)^T$. Moreover, for $k \geq M$, assume that one may choose

$$Z_k(r) = \widehat{Z}_M(r) \left(\frac{M^s}{\omega_k^s} \right),$$

where $\widehat{Z}_M(r) \in \mathbb{R}^2$ is independent of k. We justify this assumption in Section 3.2.

Definition 3.2. The $2M$ *radii polynomials* $\{p_0, p_1, \ldots, p_M\}$ *are defined by*

$$p_k(r) \stackrel{\text{def}}{=} \begin{cases} Y_k + Z_k(r) - \frac{r}{\omega_k^s} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k = 0, \ldots, M-1 \\ \widehat{Z}_M(r) - \frac{r}{\omega_M^s} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k = M. \end{cases}$$

Once these polynomials are constructed, the existence of periodic solutions to (2) can be established by using the following result, whose proof can be found in [10].

Lemma 3.3. *If there exists $r > 0$ such that $p_k(r) < 0$ for $k = 0, \ldots, M$, then there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x}$, or, equivalently, such that $f(\tilde{x}) = 0$.*

Therefore, the proofs of Theorem 1.3 and Theorem 1.4 are obtained by constructing the radii polynomials of Definition 3.2, by verifying the hypotheses of Lemma 3.3 and finally by applying Lemma 2.4. The task is now to construct the radii polynomials associated to (2).

We use the components of

$$Y_F \stackrel{\text{def}}{=} |A_M f_F(\bar{x})| \tag{21}$$

to compute Y_k for $k = 0, \ldots, M-1$. To derive the bound Z, for $b, c \in B(r)$, let $b = ur$ and $c = vr$ where $u, v \in B(1)$, and consider $A^\dagger : \Omega^{s+2} \to \Omega^s$ defined by

$$A^\dagger x \stackrel{\text{def}}{=} \begin{cases} (Df^{(M)} x_F)_k, & k = 0, \ldots M-1 \\ \Lambda_k x_k, & k \geq M, \end{cases} \tag{22}$$

an almost inverse of the operator A. Now, we have

$$DT(\bar{x} + ru)rv = [I - AA^\dagger]rv - A[Df(\bar{x} + ru) - A^\dagger]rv. \tag{23}$$

Since the first term is very small, for $k \geq 0$ and $i \in \{1, 2\}$, to derive the bound Z, we consider the expansion

$$[Df(\bar{x} + ru) - A^\dagger]rv = \sum_{l_1=1}^{4} c_{k,i}^{(l_1)} r^{l_1}. \tag{24}$$

3.1 The bounds $Z_k(r)$, $k \in \{0, \ldots, M-1\}$

For $k \in \{0, \ldots, M-1\}$, we generated coefficients $c_{k,i}^{(l_1)}$ using *Maple*. Bounds $C_k^{(l_1)} \geq 0$ such that $\left|c_{k,i}^{(l_1)}\right| \leq C_k^{(l_1)}$ can be found in Table 1. Before going any further, we make some comments on these bounds. First of all, since $c_{0,2}^{(1)} = 0$, we can define

$$C_F^{(l_1)} = \begin{pmatrix} \begin{pmatrix} C_{0,1}^{(1)} \\ 0 \end{pmatrix} \\ \begin{pmatrix} C_k^{(l_1)} \\ C_k^{(l_1)} \end{pmatrix}_{k=1,\ldots,M-1} \end{pmatrix}.$$

Thus we get that

$$|[DT(\bar{x} + ru)rv]|_F \ll |[I_M - A_M Df^{(M)}(\bar{x})]v_F| r + \sum_{l_1=1}^{4} |A_M| \cdot C_F^{(l_1)} r^{l_1}$$

Also, notice that except the infinite sum $\sum_{k_1+k_2+k_3=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s \omega_{k_3}^s}$, all other convolutions involve the finite vector $\bar{x}$ and thus are finite sums. These finite discrete convolutions are computed by using discrete fast Fourier transformation, for details on this see [14]. Finally, to bound the infinite discrete convolution in Table 1, one can use the following result.

Lemma 3.4 (**Quadratic estimates**). *For $1 \leq \mu \leq M-1$ and $\nu \in \mathbb{Z}$, let*

$$\phi_{(\mu,\nu)} = \sum_{k_1=1}^{\mu+\nu-1} \frac{1}{k_1^s(\mu+\nu-k_1)^s}$$

and

$$\alpha_{(\mu,\nu)} = \phi_{(\mu,\nu)} + 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{1}{\omega_{M+1+\mu+\nu}^s} \frac{1}{M^{(s-1)}(s-1)} + \frac{1}{\omega_{\mu+\nu}^s}\right).$$

Then

$$\sum_{k_1+k_2=\mu+\nu} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \leq \alpha_{(\mu,\nu)}. \tag{25}$$

Proof.

$$\begin{aligned}
\sum_{k_1+k_2=\mu+\nu} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} &= 2\sum_{k_1=1}^{\infty} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{2}{\omega_{\mu+\nu}^s} + \sum_{k_1=1}^{\mu+\nu-1} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu-k_1}^s} \\
&= 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \sum_{k_1=M+1}^{\infty} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{1}{\omega_{\mu+\nu}^s}\right) + \phi_{(\mu,\nu)} \\
&\leq 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{1}{\omega_{M+1+\mu+\nu}^s} \sum_{k_1=M+1}^{\infty} \frac{1}{\omega_{k_1}^s} + \frac{1}{\omega_{\mu+\nu}^s}\right) + \phi_{(\mu,\nu)} \\
&\leq 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{1}{\omega_{M+1+\mu+\nu}^s} \int_{M}^{\infty} x^{-s}dx + \frac{1}{\omega_{\mu+\nu}^s}\right) + \phi_{(\mu,\nu)} \\
&= 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{\mu+\nu+k_1}^s} + \frac{1}{\omega_{M+1+\mu+\nu}^s} \frac{1}{M^{(s-1)}(s-1)} + \frac{1}{\omega_{\mu+\nu}^s}\right) + \phi_{(\mu,\nu)}.
\end{aligned}$$

□

Remark 3.5. *To simplify our notations, whenever one of the indices is zero in a two-index notation, for instance in the case of $\phi_{(k,0)}$, we simply write ϕ_k.*

Before using (25) to formulate an explicit bound of the infinite convolution, we note that generic estimates are developed in [13] to bound higher order convolutions.

Lemma 3.6 (Cubic estimates). *For* $0 \leq k \leq M-1$,

$$
\begin{aligned}
\sum_{k_1+k_2+k_3=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s \omega_{k_3}^s} \leq & \sum_{k_1=1}^{M-k-1} \frac{1}{\omega_{k_1}^s} \alpha_{(k_1,k)} \\
& + \alpha_M \left(\frac{1}{(M-k)^s} + \frac{1}{(M-k)^{s-1}(s-1)} \right) \\
& + \phi_k + 2 \left(\sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^s \omega_{k+k_2}^s} \right. \\
& \quad \left. + \frac{1}{(M+1+k)^s M^{s-1}(s-1)} + \frac{1}{\omega_k^s} \right) \\
& + \sum_{k_1=1}^{k-1} \frac{1}{\omega_{k_1}^s} \left[\phi_{(k,-k_1)} + 2 \left(\sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^s \omega_{k-k_1+k_2}^s} \right. \right. \\
& \quad \left. \left. + \frac{1}{\omega_{M+1+k-k_1}^s} \frac{1}{M^{(s-1)}(s-1)} + \frac{1}{\omega_{k-k_1}^s} \right) \right] \\
& + 1 + \sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^{2s}} + \frac{1}{M^{2s-1}(2s-1)} \\
& + \sum_{k_1=1}^{M-1} \frac{1}{\omega_{k+k_1}^s} \alpha_{k_1} \\
& \quad + \alpha_M \left(\frac{1}{(M+k)^s} + \frac{1}{(M+k)^{(s-1)}(s-1)} \right).
\end{aligned} \tag{26}
$$

Proof.

$$
\begin{aligned}
\sum_{k_1+k_2+k_3=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s \omega_{k_3}^s} = & \sum_{k_1=1}^{\infty} \frac{1}{\omega_{k_1}^s} \sum_{k_2+k_3=k+k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} + \sum_{k_2+k_3=k} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
& + \sum_{k_1=1}^{k-1} \frac{1}{\omega_{k_1}^s} \sum_{k_2+k_3=k-k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} + \\
& + \frac{1}{\omega_k^s} \sum_{k_2+k_3=0} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
& + \sum_{k_1=1}^{\infty} \frac{1}{\omega_{k+k_1}^s} \sum_{k_2+k_3=k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s}
\end{aligned} \tag{27}
$$

Now we bound each addend in (27).

First,

$$
\begin{aligned}
\sum_{k_1=1}^{\infty} \frac{1}{\omega_{k_1}^s} \sum_{k_2+k_3=k+k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} &\leq \sum_{k_1=1}^{M-k} \frac{1}{\omega_{k_1}^s} \sum_{k_2+k_3=k+k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
&\quad + \sum_{k_1=M-k+1}^{\infty} \frac{1}{\omega_{k_1}^s} \sum_{k_2+k_3=k+k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
&\leq \sum_{k_1=1}^{M-k} \frac{1}{\omega_{k_1}^s} \alpha_{(k_1,k)} + \alpha_M \sum_{k_1=M-k+1}^{\infty} \frac{1}{\omega_{k_1}^s} \\
&\leq \sum_{k_1=1}^{M-k} \frac{1}{\omega_{k_1}^s} \alpha_{(k_1,k)} + \frac{\alpha_M}{(M-k)^{s-1}(s-1)} \qquad (28) \\
&\leq \sum_{k_1=1}^{M-k-1} \frac{1}{\omega_{k_1}^s} \alpha_{(k_1,k)} \\
&\quad + \frac{\alpha_M}{(M-k)^s} + \frac{\alpha_M}{(M-k)^{s-1}(s-1)} \\
&\leq \sum_{k_1=1}^{M-k-1} \frac{1}{\omega_{k_1}^s} \alpha_{(k_1,k)} \\
&\quad + \alpha_M \left(\frac{1}{(M-k)^s} + \frac{1}{(M-k)^{s-1}(s-1)} \right).
\end{aligned}
$$

Using Lemma 3.4, we get that

$$
\begin{aligned}
\sum_{k_2+k_3=k} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} &\leq \phi_k \\
&\quad + 2 \left(\sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^s \omega_{k+k_2}^s} + \frac{1}{(M+1+k)^s M^{s-1}(s-1)} + \frac{1}{\omega_k^s} \right). \qquad (29)
\end{aligned}
$$

Now taking $\mu = k$ and $\nu = -k_1$ in Lemma 3.4, we get that

$$
\begin{aligned}
\sum_{k_1=1}^{k-1} \frac{1}{\omega_{k_1}} \sum_{k_2+k_3=k-k_1} \frac{1}{\omega_{k_2} \omega_{k_3}} &\leq \sum_{k_1=1}^{k-1} \frac{1}{\omega_{k_1}} \left[\phi_{(k,-k_1)} + 2 \left(\sum_{k_2=1}^{M} \frac{1}{\omega_{k_2} \omega_{k-k_1+k_2}} \right.\right. \\
&\quad \left.\left. + \frac{1}{\omega_{M+1+k-k_1}} \frac{1}{M^{(s-1)}(s-1)} + \frac{1}{\omega_{k-k_1}} \right) \right]. \qquad (30)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sum_{k_2+k_3=0} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} &= 1 + 2\sum_{k_2=1}^{\infty} \frac{1}{\omega_{k_2}^{2s}} \\
&= 1 + \sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^{2s}} + \sum_{k_2=M+1}^{\infty} \frac{1}{\omega_{k_2}^{2s}} \\
&\leq 1 + \sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^{2s}} + \int_M^{\infty} x^{-2s} dx \\
&= 1 + \sum_{k_2=1}^{M} \frac{1}{\omega_{k_2}^{2s}} + \frac{1}{M^{2s-1}(2s-1)}.
\end{aligned} \tag{31}
$$

Finally, with $\mu = k_1$ and $\nu = 0$ in Lemma 3.4, we get that

$$
\begin{aligned}
\sum_{k_1=1}^{\infty} \frac{1}{\omega_{k+k_1}^s} \sum_{k_2+k_3=k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} &\leq \sum_{k_1=1}^{M} \frac{1}{\omega_{k+k_1}^s} \sum_{k_2+k_3=k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
&\quad + \sum_{k_1=M+1}^{\infty} \frac{1}{\omega_{k+k_1}^s} \sum_{k_2+k_3=k_1} \frac{1}{\omega_{k_2}^s \omega_{k_3}^s} \\
&\leq \sum_{k_1=1}^{M} \frac{1}{\omega_{k+k_1}^s} \alpha_{k_1}, + \alpha_M \sum_{k_1=M+1}^{\infty} \frac{1}{(k+k_1)^s} \\
&\leq \sum_{k_1=1}^{M} \frac{1}{\omega_{k+k_1}^s} \alpha_{k_1} + \frac{\alpha_M}{(M+k)^{(s-1)}(s-1)} \\
&\leq \sum_{k_1=1}^{M-1} \frac{1}{\omega_{k+k_1}^s} \alpha_{k_1} \\
&\quad + \frac{\alpha_M}{(M+k)^s} + \frac{\alpha_M}{(M+k)^{(s-1)}(s-1)} \\
&\leq \sum_{k_1=1}^{M-1} \frac{1}{\omega_{k+k_1}^s} \alpha_{k_1} \\
&\quad + \alpha_M \left(\frac{1}{(M+k)^s} + \frac{1}{(M+k)^{(s-1)}(s-1)} \right).
\end{aligned} \tag{32}
$$

Clearly, (28)-(32) validate the statement. □

Thus we can replace the infinite sums in Table 1 by (26). Also, the first infinite sum in Table 1 can be estimated by

$$
\sum_{k=3m-2}^{\infty} \frac{1}{\omega_k^s} \leq \frac{1}{(s-1)(3m-3)^{(s-1)}}.
$$

With these substitutions, we get new upper bounds $\mathbf{C}_F^{(l_1)}$. Now, for $k = 0, \ldots, M-1$, the k^{th}-component of $Z_k(r) \in \mathbb{R}^2$ can be defined as

$$
Z_F(r) \stackrel{\text{def}}{=} |[I_F - A_M Df^{(M)}(\bar{x})]v_F|r + \sum_{l_1=1}^{4} |A_M| \cdot \mathbf{C}_F^{(l_1)} r^{l_1}. \tag{33}
$$

3.2 The analytic bound $\hat{Z}_M(r)$

In this section, assuming $k \geq M = 3m-2$, we develop bounds $\hat{C}^{(l_i)},\ l_i \in \{1,2,3,4\}$, $i = 1,2$ satisfying

$$\left| c_{k,i}^{(l_1)} \right| \leq \frac{1}{k^{s-2}} \hat{C}^{(l_1)}.$$

Notice that these bounds depend neither on k nor i. To find these bounds, one can use

Lemma 3.7. *Let $\nu \in \mathbb{Z}$ with $k+\nu \geq M$, and $s \geq 2$, and define*

$$\gamma_{(k,\nu)} \stackrel{\text{def}}{=} 2\left(\frac{k+\nu}{k+\nu-1}\right)^s + \left[\frac{4\ln(k+\nu-2)}{k+\nu} + \frac{\pi^2-6}{3}\right]\left[\frac{2}{k+\nu} + \frac{1}{2}\right]^{s-2}$$

and

$$\alpha_{(M,\nu)} \stackrel{\text{def}}{=} \left[\frac{1}{M^2}\left\{\gamma_{(M,\nu)} + 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s} + \frac{1}{M^{s-1}(s-1)} + 1\right)\right\}\right].$$

Then

$$\sum_{k_1+k_2=k+\nu} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \leq \frac{1}{k^{(s-2)}} \alpha_{(M,\nu)}.$$

Proof. Using Lemma A.2 from [15], we get

$$\begin{aligned}
\phi_{(k,\nu)} &= \sum_{k_1=1}^{k+\nu-1} \frac{1}{k_1^s (k+\nu-k_1)^s} \\
&\leq \frac{1}{\omega_k^s}\left(2\left(\frac{k+\nu}{k+\nu-1}\right)^s + \left[\frac{4\ln(k+\nu-2)}{k+\nu} + \frac{\pi^2-6}{3}\right]\left[\frac{2}{k+\nu} + \frac{1}{2}\right]^{s-2}\right) \\
&= \frac{1}{\omega_k^s}\gamma_{(k,\nu)} \leq \frac{1}{\omega_k^s}\gamma_{(M,\nu)}
\end{aligned}$$

Now, since $k+\nu \geq M$, we get that

$$\begin{aligned}
\sum_{k_1+k_2=k+\nu} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} &\leq \frac{1}{k^s}\gamma_{(k,\nu)} + 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s \omega_{k+k_1+\nu}^s}\right. \\
&\qquad \left. + \frac{1}{(M+1+k+\nu)^s M^{s-1}(s-1)} + \frac{1}{(k+\nu)^s}\right) \\
&\leq \frac{1}{k^s}\gamma_{(M,\nu)} + \frac{2}{k^s}\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s} + \frac{1}{M^{s-1}(s-1)} + 1\right) \\
&\leq \frac{1}{k^{(s-2)}}\left[\frac{1}{M^2}\left\{\gamma_{(M,\nu)} + 2\left(\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s} + \frac{1}{M^{s-1}(s-1)} + 1\right)\right\}\right]. \square
\end{aligned}$$

Finally, we get the bound $\hat{Z}_M(r)$.

Lemma 3.8. *For $k \geq M$, define*

$$\hat{Z}_M(r) \stackrel{\text{def}}{=} \frac{1}{M^s}\left(\rho+\rho^2\left(\frac{\varepsilon \bar{L}|1-2S|}{M}+\frac{1}{M^2}\right)\right)\sum_{l_1=1}^{4} c_{k,i}^{(l_1)} r^{l_1}\begin{pmatrix}1\\1\end{pmatrix}. \tag{34}$$

Then

$$|[DT(\bar{x}+ru)rv]_k| \ll \hat{Z}_M(r)\left(\frac{M}{k}\right)^s. \tag{35}$$

Proof. Using Lemma 2.5, we get

$$\begin{aligned}
|[DT(\bar{x}+ru)rv]_k| &= \left|-\Lambda_k^{-1}[Df(\bar{x}+ru)rv - A^\dagger rv]_k\right| \\
&\ll \sum_{l_1=1}^{4}\left|-\Lambda_k^{-1}\right|\left|c_{k,i}^{(l_1)}\right| r^{l_1} \\
&\ll \sum_{l_1=1}^{4}\frac{1}{k^2}\Xi\frac{1}{k^{s-2}} r^{l_1}\hat{C}^{(l_1)}\begin{pmatrix}1\\1\end{pmatrix} r^{l_1} \\
&= \hat{Z}_M(r)\left(\frac{M}{k}\right)^s.
\end{aligned}$$

□

Now we are in the position of formulating the radii polynomials for (2). Namely, using Definition 3.2 and recalling the bounds Y_F, $Z_F(r)$ and $\hat{Z}_M(r)$ in (21), (33) and (34), respectively, we define

$$p_k(r) \stackrel{\text{def}}{=} \begin{cases} |A_M f_F(\bar{x})|_k + |[I_F - A_M Df^{(M)}(\bar{x})]v_F|r \\ \quad + \sum_{l_1=1}^{4}|A_M|\cdot \mathbf{C}_F^{(l_1)} r^{l_1} - \frac{r}{\omega_k^s}\begin{pmatrix}1\\1\end{pmatrix}, & k=0,\dots,M-1; \\ \left(\frac{1}{M^s}\left(\rho+\rho^2\left(\frac{\varepsilon\bar{L}|1-2S|}{M}+\frac{1}{M^2}\right)\right)\sum_{l_1=1}^{4} c_{k,i}^{(l_1)} r^{l_1} - \frac{r}{\omega_M^s}\right)\begin{pmatrix}1\\1\end{pmatrix}, & k=M. \end{cases}$$

4 Proofs of Theorems 1.3 and 1.4

In order to prove Theorems 1.3 and 1.4, we use the following procedure.

Procedure 4.1. *The following steps validates the assumptions of Lemma 3.3.*

1. *Fix a decay rate s and a reduction dimension m.*

2. *Find $\bar{x}_F$ an approximate solution of $f^{(m)}(x_F) = 0$. This can be done by applying a predictor-corrector continuation algorithm based on Newton's method. (See Section 4.1 for more details).*

3. *Compute the jabobian matrix $D_x f^{(M)}(\bar{x}_F)$. Compute an approximate inverse A_M of $D_x f^{(M)}(\bar{x}_F)$. This is done by using the command $A_M = \texttt{inv}(D_x f^{(M)}(\bar{x}_F)$ in MATLAB. Typically, $\|I - A_M D_x f^{(M)}(\bar{x}_F)\| \ll 1$, and hence A_M is an approximate inverse of $D_x f^{(M)}(\bar{x}_F)$. Check conditions (14) and (13) using interval arithmetic. This ensure that the linear operator A defined in (15), is invertible.*

4. *With interval arithmetic, compute the coefficients of Table 1 and 2 to complete the construction of the radii polynomials $p_1,\dots,p_M$ given in Definition 3.2.*

5. *Calculate numerically* $\mathcal{I} = (r_1^-, r_1^+) \subset \bigcap_{k=0}^{M} \{r > 0 \mid p_k(r) < 0\}$. *This step is done by computing the roots of each quartic polynomial* $p_k(r)$ *using the command* `roots` *in MATLAB. We then obtain an interval* $\mathcal{I}_k \subset \{r > 0 \mid p_k(r) < 0\}$. *Finally, we set* $\mathcal{I} \stackrel{\text{def}}{=} \bigcap_{k=0}^{M} \mathcal{I}_k$.

6. - *If* $\mathcal{I} = \varnothing$ *then go to Step 8.*
   - *If* $\mathcal{I} \neq \varnothing$ *then let* $r = \frac{r_1^- + r_1^+}{2}$. *With interval arithmetic, compute* $p_k(r)$. *If* $p_k(r) < 0$ *for all* $k = 0, \ldots, 3m-2$ *then go to Step 7; else go to Step 8.*

7. *The proof succeeds. By Lemma 3.3* T *defined in* (16) *has a unique fixed point*

$$\tilde{x} = \left(\tilde{L}, \tilde{a}_0, \tilde{a}_1, \tilde{b}_1 \tilde{a}_2, \tilde{b}_2, \tilde{a}_3, \tilde{b}_3, \ldots\right) \in \Omega^s$$

within the ball $B_{\bar{x}}(r)$. *Now, the existence of a periodic solution*

$$\tilde{y}(t) = \tilde{a}_0 + 2\sum_{k=1}^{\infty} \left[\tilde{a}_k \cos k\tilde{L}t - \tilde{b}_k \cos k\tilde{L}t\right]$$

to (2) *with* $\tilde{y}(0) = 0$ *is guaranteed by invertibility of* A *and by Lemma 2.4.*

8. *The proof fails. Either increase* $M > 3m - 2$ *or increase* m *and go to Step 2.*

4.1 Numerical solutions and proofs of Theorem 1.3 and 1.4

Consider κ as a free parameter. We isolate parameters where the trivial solution of (2) loses its stability via Hopf bifurcation. This can be done analytically see [17], [18] and [19] or using numerical tools such as TRACE-DDE, see [20]. Then with those parameters, we find a numerical approximation of a nearby periodic solution of (2). This approximation of the periodic solution is an approximate zero of

$$f(x, \kappa) = 0, \tag{36}$$

where f is defined in (9). Using this approximation as an initial point, we run the pseudo arc-length continuation algorithm of [21] on finite dimensional projection of (36) to generate other approximation of periodic solutions to (2). We then apply Procedure 4.1 to each of these numerical approximation of periodic solutions to (2).

To prove Theorem 1.3, we used $m = 37$ and $s = 2.0015$. Procedure 4.1 is then applied to each numerical approximations $\bar{x}_F$ to establish the existence of $r = r(\bar{x}_F) > 0$ such that $B_{\bar{x}_F}(r)$ contains a unique zero of (36). Now, this zero corresponds to a non-trivial periodic solution $\bar{y}(t)$ of (2). The steps are carried out in the MATLAB code *Theorem1.m* available at [23]. The code uses the interval arithmetic package INTLAB [22]. A sample approximate solution of Theorem 1.3 can be found in Table 3. The steps of the proofs are the same for Theorem 1.4, and are carried out in the MATLAB code *Theorem2.m* available at [23]. A sample approximate solution of Theorem 1.4 can be found in Table 4.

5 Future projects

As mentioned earlier, [9] considers a wide range of functional differential equations, including neutral equations. An ongoing project is to develop rigorous computational tools to study periodic solutions of equations of this type. Another project considers equations with non-polynomial nonlinearity (i.e. trigonometric nonlinearities). We are also working on developing computational techniques for state-dependent equations of threshold type relevant to modelling the dynamics of infectious diseases.

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7 Appendix

$C^{(1)}_{0,1}$	$2\sum_{k=3m-3}^{\infty}\frac{1}{\omega^s_k}$
$k\in\{0,\dots,3m-2\}.$	
$C^{(1)}_k$	$\lvert\varepsilon\,\bar L\rvert\,k\sum_{k_1+k_2+k_3=k}\left(2\,\frac{\lvert\bar a_{k_3}\bar b_{k_2}\rvert}{\omega^s_{k_1}}+\frac{\lvert\bar b_{k_3}\bar b_{k_2}\rvert}{\omega^s_{k_1}}+\frac{\lvert\bar a_{k_3}\bar a_{k_2}\rvert}{\omega^s_{k_1}}\right)$
$C^{(2)}_k$	$\frac{2\lvert\varepsilon\, k\rvert}{\omega^s_k}+\frac{4\lvert k\tau\rvert}{\omega^s_k}+2\,k^2\bar L+k^2\tau^2\left(\lvert\bar a_k\rvert+\lvert\bar b_k\rvert\right)$ $+4\,\lvert\varepsilon\,\bar L\rvert\,k\sum_{k_1+k_2+k_3=k}\frac{\lvert\bar a_{k_3}\rvert+\lvert\bar b_{k_3}\rvert}{\omega^s_{k_1}\omega^s_{k_2}}$ $+2\,\lvert\varepsilon\rvert\,k\sum_{k_1+k_2+k_3=k}\left(2\,\frac{\lvert\bar a_{k_3}\bar b_{k_2}\rvert}{\omega^s_{k_1}}+\frac{\lvert\bar b_{k_3}\bar b_{k_2}\rvert}{\omega^s_{k_1}}+\frac{\lvert\bar a_{k_3}\bar a_{k_2}\rvert}{\omega^s_{k_1}}\right)$
$C^{(3)}_k$	$\frac{3k^2}{\omega^s_k}+4\,\lvert\varepsilon\rvert\,k\sum_{k_1+k_2+k_3=k}\frac{\lvert\bar a_{k_3}\rvert+\lvert\bar b_{k_3}\rvert}{\omega^s_{k_1}\omega^s_{k_2}}$ $+4\,\lvert\varepsilon\rvert\,k\bar L\sum_{k_1+k_2+k_3=k}\frac{1}{\omega^s_{k_1}\omega^s_{k_2}\omega^s_{k_3}}$
$C^{(4)}_k$	$\frac{16\,\lvert\varepsilon\rvert\,k}{3}\sum_{k_1+k_2+k_3=k}\frac{1}{\omega^s_{k_1}\omega^s_{k_2}\omega^s_{k_3}}$

Table 1: The bounds $C^{(j)}_k$.

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$\widehat{C}_k^{(1)}$	$\frac{\varepsilon \bar{L}}{3m-2} \Bigg\{ \sum_{k_1=1}^{m-1} \Bigg[\sum_{k_2=1}^{m-1} \left\lvert \bar{a}_{k_1} \bar{b}_{k_2} \right\rvert \left(\frac{1}{\left(1 - \frac{k_1+k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} + 1 \right)$ $+ \left(\left\lvert \bar{a}_{k_1} \bar{a}_{k_2} \right\rvert + \left\lvert \bar{b}_{k_1} \bar{b}_{k_2} \right\rvert \right) \left(1 + \frac{1}{\left(1 - \frac{k_1+k_2}{3m-2}\right)^s} \right)$ $+ \sum_{k_2=k_1+1}^{m-1} \left(\left\lvert \bar{a}_{k_1} \bar{a}_{k_2} \right\rvert + \left\lvert \bar{b}_{k_1} \bar{b}_{k_2} \right\rvert \right) \left(\frac{1}{\left(1 - \frac{k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} \right)$ $+ \sum_{k_2=1}^{k_1-1} \left(\left\lvert \bar{a}_{k_1} \bar{a}_{k_2} \right\rvert + \left\lvert \bar{b}_{k_1} \bar{b}_{k_2} \right\rvert \right) \left(\frac{1}{\left(1 - \frac{k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} \right)$ $+ \left\lvert \bar{a}_0 \bar{a}_{k_1} \right\rvert \left(1 + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} \right) \Bigg] + \sum_{k_3=1}^{m-1} \left\lvert \bar{a}_{k_3} \bar{a}_0 \right\rvert \left(2 + \frac{2}{\left(1 - \frac{k_3}{3m-2}\right)^s} \right) \Bigg\}$
$\widehat{C}_k^{(2)}$	$4\bar{L} + \frac{4\lvert\tau\rvert}{(3m-2)}$ $+ \frac{\varepsilon}{(3m-2)} \Bigg\{ 2 + \sum_{k_1=1}^{m-1} \Bigg[\sum_{k_2=1}^{m-1} 4 \left\lvert \bar{a}_{k_1} \bar{b}_{k_2} \right\rvert \left(\frac{1}{\left(1 - \frac{k_1+k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} + 1 \right)$ $+2 \left(\left\lvert \bar{a}_{k_1} \bar{a}_{k_2} \right\rvert + \left\lvert \bar{b}_{k_1} \bar{b}_{k_2} \right\rvert \right) \left(\frac{1}{\left(1 - \frac{k_1+k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_2}{3m-2}\right)^s} + \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} + 1 \right) \Bigg]$ $+ \lvert\bar{L}\rvert \Bigg[\sum_{k_3=1}^{m-1} \Bigg(4 \left(\left\lvert \bar{a}_{k_3} \right\rvert + \left\lvert \bar{b}_{k_3} \right\rvert \right) \Bigg(\Bigg(\sum_{k_1=1}^{3m-2} \frac{2}{k_1^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_{(M,-k_3)} + 2 \Bigg) \frac{1}{\left(1 - \frac{k_3}{3m-2}\right)^s}$ $+ \sum_{k_1=1}^{3m-2} \frac{2}{k_1^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \Bigg)$ $+ \lvert\bar{a}_0\rvert \left(\sum_{k_1=1}^{3m-2} \frac{8}{k_1^s} + \frac{8}{(3m-2)^{s-1}(s-1)} + 4\gamma_M + 8 \right) \Bigg]$ $+ \lvert\bar{a}_0\rvert \left(\sum_{k_1=1}^{m-1} \left\lvert \bar{a}_{k_1} \right\rvert \left(4 + \frac{4}{\left(1 - \frac{k_1}{3m-2}\right)^s} \right) + 4 \right) \Bigg\}$
$\widehat{C}_k^{(3)}$	$\varepsilon \Bigg(4 \Bigg(\sum_{k_1=1}^{3m-2} \Bigg(\Bigg(\sum_{k_2=1}^{k_1-1} \frac{k_1^s}{(k_1-k_2)^s k_2^s} + \sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + 2 \Bigg) \frac{1}{k_1^s} \Bigg) + \sum_{k_2=1}^{3m-2} \frac{4}{k_2^s} + 3$ $+ \Bigg[\sum_{k_1=1}^{3m-2} \Bigg(\Bigg(\Bigg(\sum_{k_2=1}^{3m-2} \frac{1}{k_2^s} + \frac{1}{(s-1)(3m-2)^{s-1}} \Bigg) \frac{2}{k_1^s}$ $+ \sum_{k_2=1}^{k_1-1} \frac{1}{(k_1+k_2)^s k_2^s} + \frac{2}{k_1^s} \Bigg) \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} \Bigg) \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \Bigg(\sum_{k_1=1}^{3m-2} -\frac{1}{k_1^s} + \gamma_M \Bigg) \Bigg]$ $+ \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \Bigg(\sum_{k_1=1}^{3m-2} \frac{1}{k_1^s} + \frac{1}{(s-1)(3m-2)^{s-1}} \Bigg)$ $+ \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \frac{1}{(s-1)(3m-2)^{s-1}}$ $+ \frac{2}{(s-1)(3m-2)^{s-1}} + \frac{2}{(2s-1)(3m-2)^{2s-1}} + \gamma_M \Bigg) \bar{L} + \sum_{k_3=1}^{m-1} \Bigg(24 \Bigg(\Bigg(\sum_{k_1=1}^{3m-2} \frac{2}{k_1^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_{(M,-k_3)} + 2 \Bigg) \frac{1}{\left(1 - \frac{k_3}{3m-2}\right)^s}$ $+ \sum_{k_1=1}^{3m-2} \frac{2}{k_1^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \left(\left\lvert \bar{a}_{k_3} \right\rvert + \left\lvert \bar{b}_{k_3} \right\rvert \right) \Bigg)$ $+ \Bigg(\sum_{k_1=1}^{3m-2} \frac{48}{k_1^s} + \frac{48}{(s-1)(3m-2)^{s-1}} + 24\gamma_M + 48 \Bigg) \lvert\bar{a}_0\rvert \Bigg) \frac{1}{3m-2} + 3$
$\widehat{C}_k^{(4)}$	$\frac{16}{3} \Bigg(\sum_{k_1=1}^{3m-2} \Bigg(\Bigg(\sum_{k_2=1}^{k_1-1} \frac{{k_1}^s}{(k_1-k_2)^s k_2^s} + \sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + 2 \Bigg) \frac{1}{k_1^s} \Bigg)$ $+ \sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{k_2^s} + 3 + \Bigg[\sum_{k_1=1}^{3m-2} \Bigg(\Bigg(\Bigg(\sum_{k_2=1}^{3m-2} \frac{1}{k_2^s} + \frac{1}{(s-1)(3m-2)^{(s-1)}} \Bigg) \frac{2}{k_1^s} +$ $\sum_{k_2=1}^{k_1-1} \frac{1}{(k_1+k_2)^s {k_2}^s} + \frac{2}{k_1^s} \Bigg) \frac{1}{\left(1 - \frac{k_1}{3m-2}\right)^s} \Bigg) + \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \Bigg(\sum_{k_1=1}^{3m-2} -\frac{1}{k_1^s} + \gamma_M \Bigg) \Bigg]$ $+ \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \Bigg(\sum_{k_1=1}^{3m-2} \frac{1}{k_1^s} + \frac{1}{(s-1)(3m-2)^{s-1}} \Bigg)$ $+ \Bigg(\sum_{k_2=1}^{3m-2} \frac{2}{k_2^s} + \frac{2}{(s-1)(3m-2)^{s-1}} + \gamma_M + 2 \Bigg) \frac{1}{(s-1)(3m-2)^{s-1}} + \frac{2}{(s-1)(3m-2)^{s-1}} + \frac{2}{(2s-1)(3m-2)^{2s-1}} + \gamma_M \Bigg) \varepsilon$

Table 2: The bounds $\hat{C}_k^{(j)}$.

$\bar{L}$	0.588585318380744	$\bar{a}_0$	−0.000000000000000
$\bar{a}_1$	−0.380521566138099	$\bar{b}_1$	3.503923479166242
$\bar{a}_3$	0.164599872387297	$\bar{b}_3$	0.469271975833406
$\bar{a}_5$	0.120614107179449	$\bar{b}_5$	0.126650976002569
$\bar{a}_7$	0.066862971952533	$\bar{b}_7$	0.011940400142824
$\bar{a}_9$	0.024740402097032	$\bar{b}_9$	−0.010005469307534
$\bar{a}_{11}$	0.006333283564893	$\bar{b}_{11}$	−0.009564362214062
$\bar{a}_{13}$	−0.000163178672466	$\bar{b}_{13}$	−0.004751644954214
$\bar{a}_{15}$	−0.001185254556411	$\bar{b}_{15}$	−0.001658623436357
$\bar{a}_{17}$	−0.000838094624751	$\bar{b}_{17}$	−0.000288091670132
$\bar{a}_{19}$	−0.000365889550112	$\bar{b}_{19}$	0.000101927666928
$\bar{a}_{21}$	−0.000107532896479	$\bar{b}_{21}$	0.000128211215654
$\bar{a}_{23}$	−0.000006043995350	$\bar{b}_{23}$	0.000072878593114
$\bar{a}_{25}$	0.000015839170424	$\bar{b}_{25}$	0.000027885631338
$\bar{a}_{27}$	0.000012858268457	$\bar{b}_{27}$	0.000006134239246
$\bar{a}_{29}$	0.000006187458022	$\bar{b}_{29}$	−0.000001022010932
$\bar{a}_{31}$	0.000002012065977	$\bar{b}_{31}$	−0.000001933333716
$\bar{a}_{33}$	0.000000241638643	$\bar{b}_{33}$	−0.000001223207372
$\bar{a}_{35}$	−0.000000215349058	$\bar{b}_{35}$	−0.000000524561647

Table 3: Approximate zero $\hat{x}_F$ at $\tau = 2,\ \kappa = -0.0695$ and $\varepsilon = 0.15$.

$\bar{L}$	1.256637061435918	$\bar{a}_0$	0
$\bar{a}_1$	0.024710480202072	$\bar{b}_1$	−1.000002063790588
$\bar{a}_3$	−0.024488551145227	$\bar{b}_3$	−0.003663149316372
$\bar{a}_5$	−0.000266911811260	$\bar{b}_5$	0.000987238321501
$\bar{a}_7$	0.000044003554335	$\bar{b}_7$	0.000017397754331
$\bar{a}_9$	0.000001077740774	$\bar{b}_9$	−0.000002031596847
$\bar{a}_{11}$	−0.000000094942364	$\bar{b}_{11}$	−0.000000064729206
$\bar{a}_{13}$	−0.000000003804078	$\bar{b}_{13}$	−0.000000003804078
$\bar{a}_{15}$	−0.000000003804078	$\bar{b}_{15}$	0.000000000220008

Table 4: Approximate zero $\hat{x}_F$ at $\tau = 5,\ \kappa = -0.586912405465308$ and $\varepsilon = 0.25$.

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