Computational fixed point theory for differential delay equations with multiple time lags

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Abstract

We introduce a general computational fixed point method to prove existence of periodic solutions of differential delay equations with multiple time lags. The idea of such a method is to compute numerical approximations of periodic solutions using Newton's method applied on a finite dimensional projection, to derive a set of analytic estimates to bound the truncation error term and finally to use this explicit information to verify computationally the hypotheses of the Banach fixed point theorem in a given Banach space. The yielded fixed point provide us the wanted periodic solution. We provide two applications. The first one is a proof of coexistence of three periodic solutions for a given delay equation with two time lags. The second application provides a rigorous computations of several nontrivial periodic solutions for a delay equation with three time lags.

1 Introduction

Fixed point theory, the fixed point index and global bifurcation theorems are powerful tools to study the existence of solutions of infinite dimensional dynamical systems. To give a few examples in the context of functional differential equations (FDEs), the ejective fixed point theorem of Browder [1] and the fixed point index can be used to prove existence of nontrivial periodic solutions [2, 3, 4], and the global bifurcation theorem of Rabinowitz [5] can be used to prove the existence and characterize the (non) compactness of global branches of periodic solutions [6, 7]. This heavy machinery from functional analysis provides powerful existence results about solutions of FDEs, but its applicability may decrease if one asks more specific questions about the solutions of a given equation. For example, it appears difficult in general to use the ejective fixed point theorem to quantify the number of periodic solutions or to use a global bifurcation theorem to conclude about existence of folds, or more generally, of secondary bifurcations on global branches of periodic solutions. As a consequence of such limitations and with the recent availability of powerful computers and sophisticated software, numerical simulations quickly became one of the primary tool used by scientists to conjecture the answer of these explicit questions, which are often inherent to important questions in biology, physics and chemistry. For instance, in the context of population dynamics, the occurrence of a fold on a global branch of solutions may imply coexisting sub–populations.

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A standard approach adopted by mathematicians facing these explicit questions is to get insight from numerical simulations to formulate new conjectures, and then attempt to prove the conjectures using pure mathematical techniques only. As one shall argue, this strong dichotomy need not exist in the context of FDEs, as the strength of numerical analysis and functional analysis can be combined to answer, in a rigorous mathematical sense, some of the above mentioned questions regarding existence of solutions. The goal of this paper is to present such a rigorous numerical method, that we describe here as *computational fixed point theory*, to the context of proving, in a direct computational way, the existence of periodic orbits of delay differential equations with multiple time lags of the form

$$y'(t) = \mathcal{F}\left(y(t), y(t-\tau_1), \ldots, y(t-\tau_d)\right), \tag{1}$$

where $y : \mathbb{R} \to \mathbb{R}$ and $\mathcal{F} : \mathbb{R}^{d+1} \to \mathbb{R}$ is a multivariate polynomial. The idea of such method is to get numerical data from a computational algorithm (e.g. Newton's method) applied to a finite dimensional projection, to derive a set of analytic estimates to bound the truncation error term and finally to use this explicit information to verify computationally with interval arithmetic the hypotheses of a fixed point theorem (e.g. Banach fixed point theorem) in a given Banach space. The most fundamental ingredient in this process is the development of the analytic estimates, which are given in terms of regularity conditions on the solutions. As we shall see in Section 2, it turns out that the estimates for partial differential equations developed in [8, 9, 10] are perfectly suited for our context of proving existence of periodic solutions of (1).

Being both infinite dimensional dynamical systems, delay differential equations (DDEs) and partial differential equations (PDEs) share some fundamental common features. One of these features, which we strongly exploit in this work, is that bounded solutions often have more regularity than the typical solutions of the corresponding initial value problems. For instance, R. Nussbaum prove in [11] that periodic solutions of (1) are analytic by using the fact that they are bounded. This suggests that for some DDEs and PDEs the strategy of restricting our study to the set of bounded solutions can be useful, since the regularity of the solutions comes for free. Using this strategy, we define in Lemma 2.1 a Banach space $(\Omega^s, \|\cdot\|_s)$ of fast decaying coefficients which contains the Fourier coefficients of any periodic solutions of (1).

Before proceeding further, we hasten to mention that several such computational fixed point methods have been proposed to prove the existence of solutions of infinite dimensional dynamical systems (e.g. see [10, 12, 13, 14, 15, 16]). However, to the best of our knowledge this is the first attempt to integrate these techniques of rigorous computations in the context of delay equations with multiple time lags. It is also important to note that our proposed computational method is strongly influenced by the work of [13]. However, the method introduced there is specific to Wright's equation which has only one time lag. We make one more important remark before presenting explicit applications.

Remark 1.1. *The method presented in this paper should in principle be applicable to the more general class of problems of the form*

$$y^{(n+1)}(t) = \mathcal{F}\Big(y(t), y(t-\tau_1), \ldots, y(t-\tau_d), \ldots, y^{(n)}(t), y^{(n)}(t-\tau_1), \ldots, y^{(n)}(t-\tau_d)\Big)$$

where $y : \mathbb{R} \to \mathbb{R}^m$ and each components of $\mathcal{F} : \mathbb{R}^{(d+1)n} \times \mathbb{R} \to \mathbb{R}^m$ is polynomial in its first $(d+1)n$ variables. Indeed, the theory of [11] still guaranties analyticity.

Let us now introduce two applications of our computational method for DDEs with multiple delays. The first one, presented in Theorem 1.2, concerns coexistence of periodic solutions for a delay equation with two time lags and the second one, presented in Theorem 1.3, concerns existence of periodic solutions for a delay equation with three delays. The proofs are presented in Section 4.

Theorem 1.2. *The delay equations with two time lags*

$$y'(t) = -\lambda\, [y(t-\tau_1) + y(t-\tau_2)]\, [1 + y(t)], \tag{2}$$

with $\tau_1 = 1.65$ and $\tau_2 = 0.35$ has at least three nontrivial coexisting periodic solutions at the parameter value $\lambda = 3.4$. A geometric interpretation of this result can be found in Figure 1.

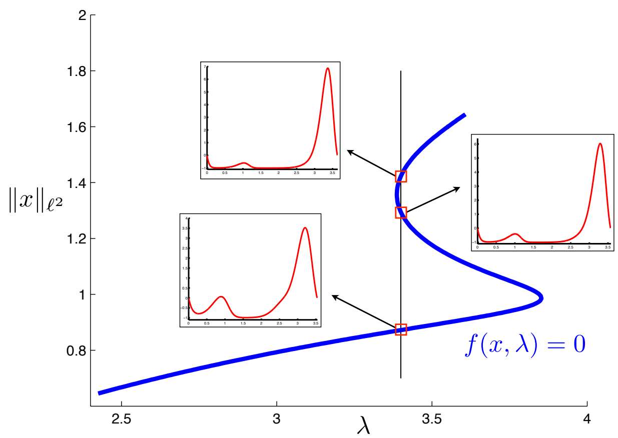

Figure 1: Coexistence of three periodic solutions for equation (2).

The proof can be found in Section 4.2.1. Concerning the result of Theorem 1.2, let us remark that in his pioneering work [17], R. Nussbaum proves the existence of at least two periodic solutions for a class of equations with two delays. However, the assumption he makes on the magnitude of λ imposes some limitation to the applicability of his method. We would like to stress that our method provides existence results without any further assumptions on the value of λ. We now present our second application.

Theorem 1.3. *Consider the differential equations with three time lags*

$$y'(t) = -\left[\lambda_1 y(t-\tau_1) + \lambda_2 y(t-\tau_2) + \lambda_3 y(t-\tau_3)\right] [1 + y(t)], \tag{3}$$

with $\lambda_1 = \lambda_2 = 2.425$, $\tau_1 = 1.65$, $\tau_2 = 0.35$ and $\tau_3 = 1$. Then, for each $\lambda_3 \in \{\frac{k}{100} \mid k = 0, \ldots, 25\}$, there exists a set B_{λ_3} containing a unique nontrivial periodic solution of (3). A geometric interpretation of this result can be found in Figure 2 and the center of the last set $B_{1/4}$ can be found in Figure 5.

The proof is presented in Section 4.2.2. Let us briefly mention about the importance of Theorem 1.3 by saying that the available information about the dynamics of equations of type (1) is very limited (e.g. see [18]). One of the most recent advancement on this field is given in [19] where the author exploited the particular properties

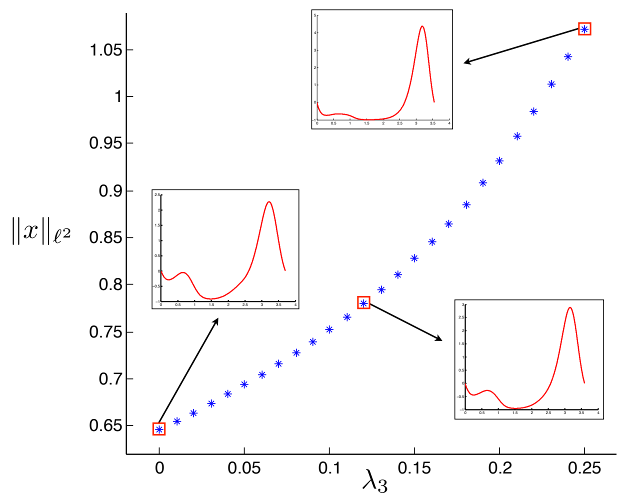

Figure 2: Proof of existence of several periodic solutions for equation (3).

of the nonlinearity to prove the existence of periodic solutions. Furthermore, many techniques available for studying the dynamics of DDEs with one delay were reviewed in [18], where it was mentioned that "*it seems that the [surveyed] techniques do not work for equations with more delays or distributed delays*". Hence, it seems reasonable to say that the study of the class of equations with many delays requires developing new techniques. We believe that already Theorem 1.2 and Theorem 1.3 give useful insight into the dynamics of this class of equations, and that our new proposed computational fixed point method can play a role in widening the knowledge about the dynamics of these equations.

The paper is organized as follows. In Section 2, we introduce an operator f whose zeros corresponds to the periodic solutions of (1). In Section 3, we transform first the problem $f(x) = 0$ into an equivalent fixed point equation $T(x) = x$. We then derive a set of sufficient computable conditions in the form of polynomial inequalities called the *radii polynomials*, whose successful verification leads to an application of the Banach fixed point theorem on T. The obtained fixed points lead to the constructive proof of existence of periodic solutions of the delay differential equation with multiple delays given by (1). In Section 4, we present applications of our method to the class of problem

$$y'(t) = -\left(\sum_{j=1}^{d} \lambda_j y(t-\tau_j)\right)[1 + y(t)], \quad (\lambda_1, \dots, \lambda_d \in \mathbb{R}),$$

where we construct explicitly the radii polynomials. We prove Theorem 1.2 and Theorem 1.3. We end the paper in Section 5 with some future projects.

2 Preliminaries

As mentioned in Section 1, the first step in the development of our computational method is to recast the problem of looking for periodic solutions of the differential equations (1) as a problem of looking for zeros of a nonlinear operator in a given Banach space $(\Omega^s, \|\cdot\|_s)$ of decaying coefficients. Let us first define the Banach space.

2.1 The Banach space Ω^s

Let $\tau_0 \stackrel{\text{def}}{=} 0$ and $y_j(t) \stackrel{\text{def}}{=} y(t-\tau_j)$ for $j = 0, \ldots, d$. Then, using the notation $\mathbf{y} \stackrel{\text{def}}{=} (y_0, y_1, \ldots, y_d)$, one can rewrite the original delay equation (1) as

$$y' = \mathcal{F}(\mathbf{y}) = \sum_{|\alpha|=1}^{N} \gamma_\alpha \mathbf{y}^\alpha, \tag{4}$$

where $\mathcal{F} : \mathbb{R}^{d+1} \to \mathbb{R}$ is a multivariate polynomial of *degree* N, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$, $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_d$ and $\mathbf{y}^\alpha = y_0^{\alpha_0} y_1^{\alpha_1} \ldots y_d^{\alpha_d}$. Periodic solutions y of (1) such that $y(t + \frac{2\pi}{L}) = y(t)$ for all $t \in \mathbb{R}$ can be expanded using the Fourier expansion

$$y(t) = \sum_{k \in \mathbb{Z}} c_k e^{ikLt}, \tag{5}$$

where the c_k are complex numbers satisfying $c_{-k} = \overline{c_k}$, since $y \in \mathbb{R}$. Denoting the real and the imaginary part of c_k respectively by a_k and b_k, one gets that $a_k = a_{-k}$ and $b_k = -b_{-k}$. As a result, $Im(c_0) = b_0 = 0$, which means that it can be neglected. An equivalent expansion for (5) is given by

$$y(t) = a_0 + 2 \sum_{k=1}^{\infty} \left[a_k \cos kLt - b_k \sin kLt \right]. \tag{6}$$

Note that one does not a priori know the frequency L of (5), meaning that we leave it as a variable. Let

$$x_k \stackrel{\text{def}}{=} \begin{cases} (L, a_0), & k = 0 \\ (a_k, b_k), & k > 0 \end{cases} \tag{7}$$

and $x \stackrel{\text{def}}{=} (x_0, x_1, \cdots, x_k, \cdots)^T$. Denote by $x_{k,1}$ and $x_{k,2}$ the first and the second component of x_k, respectively.

As mentioned earlier, a result of Nussbaum implies the analyticity of the periodic solutions of (1) (see Corollary 2 in [11]). As a consequence, the Fourier coefficients of the expansion (5), or equivalently of (6), decay faster than any algebraic decay. This important point motivates the definition of the space $(\Omega^s, \|\cdot\|_s)$. Given a growth rate $s > 0$, consider the weight functions

$$\omega_k^s = \begin{cases} 1, & k = 0; \\ k^s, & k \geq 1, \end{cases} \tag{8}$$

which are used to define the norm

$$\|x\|_s = \sup_{k=0,1,\ldots} |x_k|_\infty \omega_k^s, \tag{9}$$

where $|x_k|_\infty = \max\{|x_{k,1}|, |x_{k,2}|\}$. The proof of the following is omitted.

Lemma 2.1. *The space of sequences with algebraically decaying tails*

$$\Omega^s \stackrel{\text{def}}{=} \{x = (x_0, x_1, x_2, \ldots), \|x\|_s < \infty\} \tag{10}$$

is a Banach space. Moreover, assume that y given by (6) satisfies (1) and consider its associated x given by (7). Then for any fixed $s \geq 2$, the space Ω^s contains x.

We are now ready to introduce the operator f whose zeros in Ω^s correspond to periodic solutions of (1).

2.2 The operator equation $f(x) = 0$

Using the Fourier expansion (5), one substitutes

$$y'(t) = \sum_{k\in\mathbb{Z}} c_k ikL e^{ikLt} \quad \text{and} \quad y_j = y(t-\tau_j) = \sum_{k\in\mathbb{Z}} c_k e^{-ikL\tau_j} e^{ikLt}, \quad (j = 0, \dots, d)$$

in (4), and splitting the linear part from the nonlinear part, one gets

$$\begin{aligned}
y' - \mathcal{F}(\mathbf{y}) &= \left(y' - \sum_{|\alpha|=1} \gamma_\alpha \mathbf{y}^\alpha \right) - \sum_{|\alpha|=2}^{N} \gamma_\alpha \mathbf{y}^\alpha \\
&= \sum_{k\in\mathbb{Z}} \left(ikL - \sum_{j=0}^{d} \gamma_{e_j} e^{-ikL\tau_j} \right) c_k e^{ikLt} \\
&\quad - \sum_{|\alpha|=2}^{N} \gamma_\alpha \prod_{j=0}^{d} \left(\sum_{k_j\in\mathbb{Z}} c_{k_j} e^{-ik_jL\tau_j} e^{ik_jLt} \right)^{\alpha_j} \\
&= \sum_{k\in\mathbb{Z}} \left(ikL - \sum_{j=0}^{d} \gamma_{e_j} e^{-ikL\tau_j} \right) c_k e^{ikLt} \\
&\quad - \sum_{|\alpha|=2}^{N} \gamma_\alpha \prod_{j=0}^{d} \prod_{l=1}^{\alpha_j} \left(\sum_{k_j^{(l)}\in\mathbb{Z}} c_{k_j^{(l)}} e^{-ik_j^{(l)}L\tau_j} e^{ik_j^{(l)}Lt} \right) = 0,
\end{aligned} \tag{11}$$

where

$$\gamma_{e_0} = \gamma_{(1,0,\dots,0)}, \gamma_{e_1} = \gamma_{(0,1,0,\dots,0)}, \dots, \gamma_{e_d} = \gamma_{(0,\dots,0,1)}.$$

Note that for a smooth function y satisfying the property that $y(t + \frac{2\pi}{L}) = y(t)$ for all $t \in \mathbb{R}$, one has that the same property holds for $y' - \mathcal{F}(\mathbf{y})$, meaning that we can consider the Fourier expansion

$$y'(t) - \mathcal{F}(\mathbf{y}) = \sum_{k\in\mathbb{Z}} g_k e^{ikLt}.$$

The coefficients g_k can be computed by taking the inner product with e^{ikLt} $(k \in \mathbb{Z})$ on each side of (11). This calculation leads to

$$g_k = \left(ikL - \sum_{j=0}^{d} \gamma_{e_j} e^{-ikL\tau_j} \right) c_k - \sum_{|\alpha|=2}^{N} \gamma_\alpha \sum_{\sum_{j=0}^{d} \sum_{l=1}^{\alpha_j} k_j^{(l)} = k} \prod_{j=0}^{d} \prod_{l=1}^{\alpha_j} e^{-ik_j^{(l)}L\tau_j} c_{k_j^{(l)}}. \tag{12}$$

Hence, if y is a solution of (4) such that $y(t + \frac{2\pi}{L}) = y(t)$ for all t, then $g_k = 0$ for all $k \in \mathbb{Z}$. It is not hard to realize that since $c_{-k} = \overline{c_k}$, then one has that $g_{-k} = \overline{g_k}$. As a consequence, one needs only to consider the cases $k \geq 0$ when solving for $g_k = 0$. As in the case of c_0, one also gets that $Im(g_0) = 0$. In order to eliminate arbitrary shifts, we impose a normalizing condition $y(0) = \psi_0 \in \mathbb{R}$. The value of ψ_0 is chosen depending on the problem. For instance, if we a priori know that we are looking for periodic solutions oscillating around zero, one can let $\psi_0 = 0$. Hence, we are looking for $x \in \Omega^s$ such that the following holds

$$h(x) \stackrel{\text{def}}{=} y(0) - \psi_0 = a_0 + 2\sum_{k=1}^{\infty} a_k - \psi_0 = 0, \tag{13}$$

and such that $Re(g_k)(x) = 0$ and $Im(g_k)(x) = 0$. The next step is to compute the real and the imaginary parts of g_k. Given $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \mathbb{N}^{d+1}$, consider the coefficients $\sigma^{(r)}_{\mu,\nu}, \sigma^{(i)}_{\mu,\nu} \in \{-1, 0, 1\}$ indexed over $\mu, \nu \in \{0,1\}^{|\alpha|}$ with $|\mu| + |\nu| = |\alpha|$ such that the following expansion holds

$$\prod_{j=0}^{d}\prod_{l=1}^{\alpha_j} e^{-ik_j^{(l)}L\tau_j} c_{k_j^{(l)}} = e^{-i\xi L}\prod_{j=0}^{d}\prod_{l=1}^{\alpha_j} c_{k_j^{(l)}} = e^{-i\xi L}\left(\sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(r)}_{\mu,\nu} a^\mu b^\nu + i \sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(i)}_{\mu,\nu} a^\mu b^\nu\right)$$

where

$$\begin{aligned}
a &= \left(a_{k_0^{(1)}}, \dots, a_{k_0^{(\alpha_0)}}, \dots, a_{k_d^{(1)}}, \dots, a_{k_d^{(\alpha_d)}}\right) \in \mathbb{R}^{|\alpha|} \\
b &= \left(b_{k_0^{(1)}}, \dots, b_{k_0^{(\alpha_0)}}, \dots, b_{k_d^{(1)}}, \dots, b_{k_d^{(\alpha_d)}}\right) \in \mathbb{R}^{|\alpha|} \\
\mu &= \left(\mu_{k_0^{(1)}}, \dots, \mu_{k_0^{(\alpha_0)}}, \dots, \mu_{k_d^{(1)}}, \dots, \mu_{k_d^{(\alpha_d)}}\right) \in \{0,1\}^{|\alpha|} \\
\nu &= \left(\nu_{k_0^{(1)}}, \dots, \nu_{k_0^{(\alpha_0)}}, \dots, \nu_{k_d^{(1)}}, \dots, \nu_{k_d^{(\alpha_d)}}\right) \in \{0,1\}^{|\alpha|} \\
\xi &= \sum_{j=0}^{d} (k_j^{(1)} + \cdots + k_j^{(\alpha_j)})\tau_j.
\end{aligned}$$

Using the above expansion, one has that

$$\begin{aligned}
Re(g_k)(x) &= \left(-\sum_{j=0}^{d}\gamma_{e_j}\cos kL\tau_j\right) a_k + \left(-kL - \sum_{j=0}^{d}\gamma_{e_j}\sin kL\tau_j\right) b_k \qquad (14) \\
&\quad - \sum_{|\alpha|=2}^{N}\gamma_\alpha \sum_{\sum_{j=0}^{d}\sum_{l=1}^{\alpha_j} k_j^{(l)} = k} \left(\cos\xi L \sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(r)}_{\mu,\nu} a^\mu b^\nu + \sin\xi L \sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(i)}_{\mu,\nu} a^\mu b^\nu\right) \\
Im(g_k)(x) &= \left(kL + \sum_{j=0}^{d}\gamma_{e_j}\sin kL\tau_j\right) a_k + \left(-\sum_{j=0}^{d}\gamma_{e_j}\cos kL\tau_j\right) b_k \qquad (15) \\
&\quad - \sum_{|\alpha|=2}^{N}\gamma_\alpha \sum_{\sum_{j=0}^{d}\sum_{l=1}^{\alpha_j} k_j^{(l)} = k} \left(-\sin\xi L \sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(i)}_{\mu,\nu} a^\mu b^\nu + \cos\xi L \sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu\in\{0,1\}^{|\alpha|}}} \sigma^{(r)}_{\mu,\nu} a^\mu b^\nu\right).
\end{aligned}$$

As mentioned in Section 1, the most fundamental ingredient in developing our computational fixed point method is to derive a set of analytic estimates to bound the truncation error terms. The following estimates are given in terms of regularity conditions on the solutions and they were originally developed to prove constructively existence of equilibria of PDEs (see [8, 9, 10]).

Lemma 2.2 (Analytic Estimates)**.** *Fix a decay rate $s \geq 2$ and assume that there exist asymptotic constants A_s and B_s such that the following regularity conditions hold*

$$|a_k| \leq \frac{A_s}{\omega_k^s} \quad \text{and} \quad |b_k| \leq \frac{B_s}{\omega_k^s}, \quad \text{for all } k \in \mathbb{Z}.$$

Consider $k \geq 0$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$ and $\mu, \nu \in \{0,1\}^{|\alpha|}$ with $|\mu| + |\nu| = |\alpha|$. Then there exists an explicit and constant $C = C(A_s, B_s) < \infty$ which is independent of k, such that

$$\left| \sum_{\sum_{j=0}^{d} \sum_{l=1}^{\alpha_j} k_j^{(l)} = k} a^\mu b^\nu \right| \leq \frac{C}{\omega_k^s}. \tag{16}$$

Proof. See Lemma 2.1 in [8]. □

Lemma 2.3 (Definition of f)**.** *The operator $f = \{f_k\}_{k \geq 0}$ whose k^{th} component $f_k \in \mathbb{R}^2$ is given by*

$$f_k(x) \stackrel{\text{def}}{=} \begin{cases} \begin{pmatrix} h(x) \\ Re(g_0)(x) \end{pmatrix}, & k = 0 \\ \begin{pmatrix} Re(g_k)(x) \\ Im(g_k)(x) \end{pmatrix}, & k > 0 \end{cases} \tag{17}$$

is such that $f : \Omega^s \to \Omega^{s-1}$.

Proof. Consider $x \in \Omega^s$. Then using the definition of Ω^s in (10), one can easily verify that

$$\sup_{k \geq 0} \left| \left(-\sum_{j=0}^{d} \gamma_{e_j} \cos kL\tau_j \right) a_k + \left(-kL - \sum_{j=0}^{d} \gamma_{e_j} \sin kL\tau_j \right) b_k \right| \omega_k^{s-1} < \infty$$
$$\sup_{k \geq 0} \left| \left(kL + \sum_{j=0}^{d} \gamma_{e_j} \sin kL\tau_j \right) a_k + \left(-\sum_{j=0}^{d} \gamma_{e_j} \cos kL\tau_j \right) b_k \right| \omega_k^{s-1} < \infty.$$

Now, since $x \in \Omega^s$, then $\|x\|_s < \infty$, and since $x_{k,1} = a_k$, $x_{k,2} = b_k$, one has that

$$|a_k| \leq \frac{\|x\|_s}{\omega_k^s} \quad \text{and} \quad |b_k| \leq \frac{\|x\|_s}{\omega_k^s}, \quad \text{for all } k \in \mathbb{Z}.$$

Consider now $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$ and $\mu, \nu \in \{0,1\}^{|\alpha|}$ with $|\mu| + |\nu| = |\alpha|$. Then, by Lemma 2.2, one can conclude about the existence of $C = C(\|x\|_s) < \infty$ such that

$$\sup_{k \geq 0} \left| \sum_{\sum_{j=0}^{d} \sum_{l=1}^{\alpha_j} k_j^{(l)} = k} a^\mu b^\nu \right| \omega_k^s \leq C < \infty.$$

Finally, using the above inequalities, one obtains

$$\|f(x)\|_{s-1} = \sup_{k \geq 0} |f_k|_\infty \omega_k^{s-1} < \infty,$$

and this shows that $f(x) \in \Omega^{s-1}$. □

As in the case of x, denote by $f_{k,1}$ and $f_{k,2}$ the first and the second component of f_k respectively. For the sake of simplicity of the presentation, let us introduce a more compact representation of the operator $f : \Omega^s \to \Omega^{s-1}$. Defining

$$R_k(L) \stackrel{\text{def}}{=} \begin{pmatrix} -\displaystyle\sum_{j=0}^{d} \gamma_{e_j} \cos kL\tau_j & -\left(kL + \displaystyle\sum_{j=0}^{d} \gamma_{e_j} \sin kL\tau_j\right) \\ kL + \displaystyle\sum_{j=0}^{d} \gamma_{e_j} \sin kL\tau_j & -\displaystyle\sum_{j=0}^{d} \gamma_{e_j} \cos kL\tau_j \end{pmatrix} \tag{18}$$

$$\Theta(L) \stackrel{\text{def}}{=} \begin{pmatrix} \cos \xi L & \sin \xi L \\ -\sin \xi L & \cos \xi L \end{pmatrix}, \tag{19}$$

one obtains that for $k \geq 1$

$$f_k(x) = R_k(L) \begin{pmatrix} a_k \\ b_k \end{pmatrix} - \sum_{|\alpha|=2}^{N} \gamma_\alpha \sum_{\sum_{j=0}^{d} \sum_{l=1}^{\alpha_j} k_j^{(l)} = k} \Theta(L) \begin{pmatrix} \displaystyle\sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu \in \{0,1\}^{|\alpha|}}} \sigma_{\mu,\nu}^{(r)} a^\mu b^\nu \\ \displaystyle\sum_{\substack{|\mu|+|\nu|=|\alpha| \\ \mu,\nu \in \{0,1\}^{|\alpha|}}} \sigma_{\mu,\nu}^{(i)} a^\mu b^\nu \end{pmatrix}. \tag{20}$$

The following results provides an equivalence between the solutions $y(t)$ of (1) satisfying $y(t) = y(t + \frac{2\pi}{L})$ and the solutions $x \in \Omega^s$ of $f = 0$, where f is given componenet-wise by (17).

Lemma 2.4. *A sequence $x = (x_0, x_1, x_2, \dots) \in \Omega^s$ given by (7) is such that $f(x) = 0$ if and only if $y(t)$ given by (6) solves the equation (1) such that $y(0) = \psi_0$.*

Proof. Using the analytic estimates of Lemma 2.2, the proof is similar to the proof of Lemma 3.2(b) in [13]. □

3 Computational fixed point method

The first part of this section is to transform the problem $f(x) = 0$ into an equivalent fixed point equation $T(x) = x$. The second part is to derive a set of sufficient computable conditions called the *radii polynomials*, whose successful verification leads to an application of the Banach fixed point theorem on T. The obtained fixed points lead to the constructive proof of existence of periodic solutions of the delay differential equation with multiple delays given by (1). Note that the name *computational fixed point theory* comes from the fact that the verification of the hypotheses of the Banach fixed point theorem in Lemma 3.2 is done computationally via the radii polynomials of Definition 3.3.

3.1 The fixed point operator equation $T(x) = x$

Since we want to develop this idea in a computational setting, consider a finite dimensional projection $f^{(m)} : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ of f, defined component-wise by

$$f_k^{(m)}(x_0, \dots, x_{m-1}) \stackrel{\text{def}}{=} f_k\left(x_0, \dots, x_{m-1}, 0_\infty\right), \ k = 0, \dots, m-1, \tag{21}$$

where $0_\infty = (0)_{j \geq 0}$. Suppose that using a Newton-like iterative scheme, we computed numerically $\bar{x} \in \mathbb{R}^{2m}$ such that $f^{(m)}(\bar{x}) \approx 0$. To simplify the presentation, we identify

$\bar{x} = (\bar{L}, \bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_{m-1}, \bar{b}_{m-1})^T$ with $(\bar{x}, 0_\infty)$. The philosophy behind the construction of the fixed point equation $T(x) = x$ is to define its finite part in Ω^s with the use of numerics and to define its tail in Ω^s analytically. Consider a computational parameter $M \geq m$ which we describe in details in Section 3.2. In order to define the finite part of T consider a $2M$ by $2M$ matrix A_M which is a numerical approximation of the inverse of $Df^{(M)}(\bar{x})$. In order to define the tail part ot T, one defines

$$\Lambda_k \stackrel{\text{def}}{=} \frac{\partial f_k}{\partial x_k}(\bar{x}) = \begin{pmatrix} \frac{\partial f_{k,1}}{\partial x_{k,1}}(\bar{x}) & \frac{\partial f_{k,1}}{\partial x_{k,2}}(\bar{x}) \\ \frac{\partial f_{k,2}}{\partial x_{k,1}}(\bar{x}) & \frac{\partial f_{k,2}}{\partial x_{k,2}}(\bar{x}) \end{pmatrix}. \tag{22}$$

Note that since the matrices given by (18) and (19) are skew-symmetric, it follows from (20) that, for $k \geq 1$, the matrix Λ_k is also skew-symmetric. More explicitly, using the definition of $R_k(L)$ defined in (18), one gets that

$$\Lambda_k = \begin{pmatrix} \tau_k & \delta_k \\ -\delta_k & \tau_k \end{pmatrix},$$

where $\tau_k, \delta_k \in \mathbb{R}$ and

$$\delta_k = -kL + \zeta_k, \tag{23}$$

with $\zeta_k \in \mathbb{R}$. The idea behind the definition of the fixed point equation $T(x) = x$ is that it is a Newton-like operator on the function space Ω^s of the form $T(x) = x - Af(x)$, where A is an approximation for $Df(\bar{x})$. The following result defines explicitly the linear operator A. For a given $x \in \Omega^s$, denote by x_F the finite dimensional projection $x_F = (x_0, x_1, \ldots, x_{M-1}) \in \mathbb{R}^{2M}$.

Lemma 3.1. *Consider $\bar{x} = (\bar{L}, \bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_{m-1}, \bar{b}_{m-1})$ a numerical approximation of the Galerkin approximation (21). Assume that*

$$\|A_M D_x f^{(M)}(\bar{x}) - I_M\|_\infty < 1, \tag{24}$$

where I_M is the $2M \times 2M$ identity matrix. Suppose the existence of two constants $\zeta^*, \tau^* \geq 0$ such that

$$|\zeta_k| \leq \zeta^* < M\bar{L} \quad \text{and} \quad |\tau_k| \leq \tau^* \quad \text{for all} \quad k \geq M, \tag{25}$$

and assume that

$$\delta_k < 0, \quad \text{for all} \quad k \geq M. \tag{26}$$

Then, the matrix Λ_k given by (22) is invertible for all $k \geq M$ and the linear operator $A : \Omega^s \to \Omega^{s+1}$ defined by

$$Ax \stackrel{\text{def}}{=} \begin{cases} (A_M x_F)_k, & k = 0, \ldots, M-1, \\ \Lambda_k^{-1} x_k, & k \geq M \end{cases} \tag{27}$$

is invertible.

Proof. First of all, using condition (26), one has that for $k \geq M$, $\det(\Lambda_k) = \tau_k^2 + \delta_k^2 > 0$ and therefore Λ_k is invertible for any $k \geq M$. By (24), one gets that the matrix A_M is invertible, which then implies that A is invertible. It remains to show that A maps Ω^s into Ω^{s+1}. For this, one needs to describe the asymptotic behavior of $\{\Lambda_k^{-1} x_k\}_{k \geq M}$. Let us define the number

$$\rho \stackrel{\text{def}}{=} \frac{M}{M\bar{L} - \zeta^*} > 0 \tag{28}$$

which is positive by condition (25), and the matrix

$$\Xi \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\rho^2\tau^*}{M} & \rho \\ \rho & \frac{\rho^2\tau^*}{M} \end{pmatrix}. \tag{29}$$

Combining (23), (25) and (26), one has that for $k \geq M$, $|\delta_k| = k\bar{L} - \zeta_k \geq k\bar{L} - \zeta^* = k\left(\bar{L} - \frac{\zeta^*}{k}\right) \geq k\left(\bar{L} - \frac{\zeta^*}{M}\right) = \frac{k}{\rho} > 0$. Therefore, one gets that $\left|\frac{\delta_k}{\tau_k^2+\delta_k^2}\right| \leq \frac{|\delta_k|}{\delta_k^2} = \frac{1}{|\delta_k|} \leq \frac{1}{k}\rho$. Also, since $|\tau_k| \leq \tau^*$, one gets that $\left|\frac{\tau_k}{\tau_k^2+\delta_k^2}\right| \leq \frac{\tau^*}{\delta_k^2} \leq \frac{\rho^2\tau^*}{k^2} \leq \frac{1}{k}\left(\frac{\rho^2\tau^*}{M}\right)$. As a result, one conclude that component-wise,

$$\left|{\Lambda_k}^{-1}\right| = \frac{1}{\tau_k^2+\delta_k^2}\begin{pmatrix} |\tau_k| & |\delta_k| \\ |\delta_k| & |\tau_k| \end{pmatrix} \leq_{cw} \frac{1}{k}\Xi.$$

Let us now consider $x \in \Omega^s$. Then, recalling (9) and (27), one has that

$$\begin{aligned} \|Ax\|_{s+1} &= \max\left\{\max_{k=0,\dots,M-1}\{|(A_M x_F)_k|_\infty \omega_k^{s+1}\}\ ,\ \sup_{k\geq M}\{|\Lambda_k^{-1}x_k|_\infty \omega_k^{s+1}\}\right\} \\ &\leq \max\left\{\max_{k=0,\dots,M-1}\{|(A_M x_F)_k|_\infty \omega_k^{s+1}\}\ ,\ \|\Xi\|_\infty \sup_{k\geq M}\{|x_k|_\infty \omega_k^{s}\}\right\} < \infty. \end{aligned}$$

□

We are now ready to define the fixed point equation. Combining the results of Lemma 2.3 and Lemma 3.1, one can define the nonlinear Newton-like operator $T : \Omega^s \to \Omega^s$ by

$$T(x) \stackrel{\text{def}}{=} x - A \cdot f(x) \tag{30}$$

It is important to remark that even if we construct the finite part of the operator T in a computer-assisted fashion, we still think of if as an abstract object. The finite part is stored on a computer, and the tail part, consisting of the sequence of matrices $\{\Lambda_k^{-1}\}_{k\geq M}$, is defined abstractly. Also, it is important to remark that the invertibility of the operator A implies that the fixed points of T given by (30) are in bijection with the zeros of f given by (17). Let us now introduce the notion of radii polynomials in order to verify in a computationally efficient way the hypotheses of the Banach fixed point theorem applied to the map T.

3.2 The radii polynomials

We have shown in Lemma 2.4 that the problem of finding solutions y of (1) such that $y(t) = y(t + \frac{2\pi}{L})$ and $y(0) = \psi_0$ is the same than studying the zeros of f, or equivalently the fixed points of T. We now turn to the problem of deriving a set of computable conditions called the *radii polynomials*, whose successful verification leads to an application of a contraction mapping argument. The idea of such an approach is to consider balls of the form $B_{\bar{x}}(r) \in \Omega^s$ centered at the numerical solution $\bar{x}$ of unknown radii r, and to solve for the radius r for which $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping. It is important to note that the idea of the radii polynomials is not new and was first introduced in the context of proving existence of equilibria of one dimensional PDEs [10, 20]. This concept was then adapted to prove existence of periodic solutions of ordinary differential equations (ODEs) [9], of periodic solutions of DDEs with one time lag [13], of equilibria of higher dimensional PDEs [8, 21] and of connecting orbits of ODEs [22]. We now introduce them in the context of DDEs with multiple time lags.

Consider the ball of radius r in Ω^s (with norm $\|\cdot\|_s$), centered at the origin,

$$B(r) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} \left[-\frac{r}{\omega_k^s}, \frac{r}{\omega_k^s}\right]^2 \tag{31}$$

and the balls centered at $\bar{x}$

$$B_{\bar{x}}(r) = \bar{x} + B(r). \tag{32}$$

Let us introduce the notation of component-wise inequality by $\leq_{cw}$, which means that for $u, v \in \mathbb{R}^{m\times n}$ one has that $u \leq_{cw} v$ if $u_{i,j} \leq v_{i,j}$, for all $i = 1, \dots, m$ and $j = 1, \dots, n$. To show that T is a contraction mapping, we need component-wise positive bounds $Y_k = \binom{Y_{k,1}}{Y_{k,2}}$, $Z_k = \binom{Z_{k,1}}{Z_{k,2}} \in \mathbb{R}^2$ for each $k \geq 0$, such that

$$\Big|[T(\bar{x}) - \bar{x}]_k\Big| = \Big|[-A \cdot f(\bar{x})]_k\Big| \leq_{cw} Y_k, \tag{33}$$

and

$$\sup_{b,c\in B(r)} \Big|[DT(\bar{x} + b)c]_k\Big| \leq_{cw} Z_k(r). \tag{34}$$

Lemma 3.2. *Fix $s \geq 2$ and consider the bounds $Y = (Y_0, Y_1, \dots)$ and $Z = (Z_0, Z_1, \dots)$ satisfying (33) and (34). If there exists an $r > 0$ such that $\|Y + Z\|_s < r$, then the operator T given by* (30) *maps the ball $B_{\bar{x}}(r)$ into itself and $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. By the Banach Fixed Point Theorem, there is a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x}$, or equivalently such that $f(\tilde{x}) = 0$. Moreover, a rigorous upper bound for the ℓ_2–error between the numerical approximation $\bar{x}$ and the solution $\tilde{x}$ is given by*

$$\|\tilde{x} - \bar{x}\|_{\ell^2} \leq r\sqrt{4 + \frac{2}{2s-1}}. \tag{35}$$

Proof. For the existence of the $\tilde{x} \in B_{\bar{x}}(r)$ such that $f(\tilde{x}) = 0$, we refer to [13]. For the remaining part of the proof, it is sufficient to observe that $\tilde{x} - \bar{x} \in B(r)$ and that

$$\begin{aligned}
\|\tilde{x} - \bar{x}\|_{\ell^2} &= \sqrt{\sum_{k=0}^{\infty} \Big((\tilde{a}_k - \bar{a}_k)^2 + (\tilde{b}_k - \bar{b}_k)^2\Big)} \leq \sqrt{\sum_{k=0}^{\infty} \left(\frac{r}{\omega_k^s}\right)^2 + \left(\frac{r}{\omega_k^s}\right)^2} \\
&\leq r\sqrt{\sum_{k=0}^{\infty} \frac{2}{\omega_k^{2s}}} \leq r\sqrt{4 + \int_1^{\infty} \frac{2}{x^{2s}} ds} = r\sqrt{4 + \frac{2}{2s-1}}.
\end{aligned}$$

□

Before introducing the notion of the radii polynomials, let us be more explicit about the computational parameter M used to define the linear operator A given by (27). This parameter is chosen so that one may chose $Y_k = \binom{0}{0}$ for all $k \geq M$. Recalling the order N of the multivariate polynomial $\mathcal{F}$ given by (4), let us define

$$M \stackrel{\text{def}}{=} N(m-1) + 1. \tag{36}$$

Hence, for any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \mathbb{N}^{d+1}$ such that $|\alpha| \leq N$ and $\mu, \nu \in \{0,1\}^{|\alpha|}$ such that $|\mu| + |\nu| = |\alpha|$, one can use the fact that $\bar{a}_k = \bar{b}_k = 0$ for all $k \geq m$ to conclude that

$$\sum_{\sum_{j=0}^{d} \sum_{l=1}^{\alpha_j} k_j^{(l)} = k} \bar{a}^{\mu} \bar{b}^{\nu} = 0, \quad \text{for all} \quad k \geq M.$$

Therefore, for $k \geq M$, one can see from (20) that $f_k(\bar{x}) = (0,0)^T$ and by definition of A for $k \geq M$ given in (27), one gets that $[-A \cdot f(\bar{x})]_k = \Lambda_k^{-1} f_k(\bar{x}) = (0,0)^T$. Hence, by the choice of M given by (36), we can define $Y_k = \binom{0}{0}$, for $k \geq M$. Note that corresponding to the entries $i \in \{1,2\}$ and $k \in \{0, \dots, M-1\}$ of Y one defines

$$Y_F \stackrel{\text{def}}{=} |A_M \cdot f_F(\bar{x})| \,. \tag{37}$$

The above discussion clarifies the choice of the computational parameter M and the computation of the bound Y given by (33). The computation of the bound Z given by (34) is more involved and rather than computing it in the general context, we refer to Section 4.1.1 and Section 4.1.2 where it is constructed explicitly in the context of a given class of problems. At the moment, it is sufficient to understand that each $Z_k(r)$ can be computed as a polynomial in the a priori unknown radius r. As we see in Section 4.1, the idea is to factor the points $b, c \in B(r)$ as $b = ur$ and $c = vr$ with $u, v \in B(1)$, to expand in the unknown variable radius r, and finally to use the fact that $u, v \in B(1)$ to compute uniform bounds in u, v. We are ready to define the radii polynomials.

Definition 3.3. *For $k \geq M$, let $Y_k = \binom{0}{0}$ and assume that one may chose $Z_k(r) = \hat{Z}_M(r) \left(\frac{M^s}{\omega_k^s}\right)$, where $\hat{Z}_M(r) >_{cw} \binom{0}{0}$ is independent of k. We define the $2M+2$* radii polynomials *$\{p_0, \dots, p_{M-1}, p_M\}$ by*

$$p_k(r) \stackrel{\text{def}}{=} \begin{cases} Y_k + Z_k(r) - \frac{r}{\omega_k^s} \binom{1}{1}, & k = 0, \dots, M-1; \\ \hat{Z}_M(r) - \frac{r}{\omega_M^s} \binom{1}{1} & k = M. \end{cases}$$

We refer to Lemma 4.4 for an explicit example of how to derive the bounds $Z_k(r) = \hat{Z}_M(r) \left(\frac{M^s}{\omega_k^s}\right)$ for $k \geq M$.

Lemma 3.4. *If there exists an $r > 0$ such that $p_k(r) < 0$ for all $k = 0, \dots, M$, then there exist a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x}$, or equivalently such that $f(\tilde{x}) = 0$.*

Proof. See [13]. □

Our computational fixed point method then consist of constructing the radii polynomials, verify (if possible) the hypotheses of Lemma 3.4 and finally apply Lemma 2.4 to conclude about the constructive proof of existence of periodic solutions of (1). We are now ready for some applications.

4 Applications

In this section, we focus our study on the class of equations of the form

$$y'(t) = -\left(\sum_{j=1}^{d} \lambda_j y(t - \tau_j)\right) [1 + y(t)], \quad (\lambda_1, \dots, \lambda_d \in \mathbb{R}), \tag{38}$$

which falls in the category of (1). The essential step in obtaining constructive proofs of existence of nontrivial periodic solutions for (38) is to construct the radii polynomials of Definition 3.3 and then to apply Lemma 3.4. In Section 4.1, we provide an explicit construction of the radii polynomials in the context of (38) and in Section 4.2, we provide the proofs of Theorem 1.2 and of Theorem 1.3.

4.1 Radii polynomials for the class of equations (38)

First of all, let us observe that in the context of problem (38), the expansions (14) and (15) are given respectively by

$$Re(g_k)(x) = \left(\sum_{j=1}^{d} \lambda_j \cos kL\tau_j\right) a_k + \left(-kL + \sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right) b_k$$

$$+ \sum_{k_1+k_2=k} \left(\sum_{j=1}^{d} \lambda_j \cos kL\tau_j\right) (a_{k_1}a_{k_2} - b_{k_1}b_{k_2}) + \left(\sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right) (a_{k_1}b_{k_2} + b_{k_1}a_{k_2})$$

$$Im(g_k)(x) = -\left(-kL + \sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right) a_k + \left(\sum_{j=1}^{d} \lambda_j \cos kL\tau_j\right) b_k$$

$$+ \sum_{k_1+k_2=k} -\left(\sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right) (a_{k_1}a_{k_2} - b_{k_1}b_{k_2}) + \left(\sum_{j=1}^{d} \lambda_j \cos kL\tau_j\right) (a_{k_1}b_{k_2} + b_{k_1}a_{k_2}).$$

Let us restrict our study of (38) to nontrivial periodic solutions oscillating around the trivial solution $y = 0$. Hence, the scalar quantity ψ_0 used to eliminate arbitrary time shift in equation (13) is set to be $\psi_0 = 0$. In the context of (38), the *degree* of the multivariate polynomial (4) is $N = 2$, which means that the computational parameter M given by (36) is set to be $M = 2m - 1$. Also, the matrix Λ_k $(k \geq M)$ given by (22) used to define the tail of the operator A in (27) is given by

$$\Lambda_k = \begin{pmatrix} \tau_k & \delta_k \\ -\delta_k & \tau_k \end{pmatrix}, \tag{39}$$

where $\tau_k \overset{\text{def}}{=} \left(\sum_{j=1}^{d} \lambda_j\right) \bar{a}_0 + (1 + \bar{a}_0) \left(\sum_{j=1}^{d} \lambda_j \cos kL\tau_j\right)$ and $\delta_k \overset{\text{def}}{=} -k\bar{L} + (1 + \bar{a}_0) \left(\sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right)$. One can verify that $\zeta^*, \tau^* \geq 0$ satisfying (25) can be chosen to be

$$\zeta^* \overset{\text{def}}{=} |1 + \bar{a}_0| \sum_{j=1}^{d} |\lambda_j| \quad \text{and} \quad \tau^* \overset{\text{def}}{=} (|1 + \bar{a}_0| + |\bar{a}_0|) \sum_{j=1}^{d} |\lambda_j|. \tag{40}$$

The following result provides an explicit lower bound on the projection dimension m so that the matrices given by (22) are invertible for all $k \geq M = 2m - 1$.

Lemma 4.1. *Let m be the dimension of the finite dimensional reduction (21) and let $M = 2m - 1$. If*

$$m > \frac{1}{2} \left[\frac{|1 + \bar{a}_0|}{\bar{L}} \left(\sum_{j=1}^{d} |\lambda_j|\right) + 1 \right]. \tag{41}$$

then $\delta_k < 0$ for all $k \geq M = 2m - 1$ and $\zeta^* < M\bar{L}$. Hence, the matrix Λ_k given in (22) is invertible for all $k \geq M$.

Proof. For $k \geq M$, $(1+\bar{a}_0)\left(\sum_{j=1}^{d} \lambda_j \sin kL\tau_j\right) \leq \zeta^* = |1+\bar{a}_0|\left(\sum_{j=1}^{d} |\lambda_j|\right) < M\bar{L} \leq k\bar{L}$, which follows from condition (41). Hence, we get that $\delta_k < 0$ for all $k \geq M$ and we can conclude that $det(\Lambda_k) = \tau_k^2 + \delta_k^2 > 0$, for all $k \geq M$. □

In order to apply Lemma 3.4, one needs to construct the radii polynomials given by Definition 3.3. Their construction requires the computation of the bounds Y and Z given respectively by (33) and (34). The nontrivial $Y_0, \ldots, Y_{M-1}$ can be computed using (37). Therefore, the remaining ingredients to finalize the construction of the polynomials in the context of (38) is the bound Z satisfying

$$\sup_{b,c\in B(r)} \left| [DT(\bar{x}+b)c]_k \right| = \sup_{u,v\in B(1)} \left| [DT(\bar{x}+ru)rv]_k \right| \leq_{cw} Z_k(r),$$

where as mentioned earlier, the idea is to factor the points $b, c \in B(r)$ as $b = ur$ and $c = vr$ with $u, v \in B(1)$, to expand in the unknown variable radius r, and finally to use the fact that $u, v \in B(1)$ to compute uniform bounds in u, v. Introducing an almost inverse $A^\dagger : \Omega^{s+1} \to \Omega^s$ of the operator A defined in (27)

$$A^\dagger x \stackrel{\text{def}}{=} \begin{cases} (Df^{(M)} x_F)_k, & k = 0, \ldots, M-1, \\ \Lambda_k x_k, & k \geq M \end{cases}$$

we can split $Df(\bar{x}+ru)rv = A^\dagger rv + \left[Df(\bar{x}+ru)rv - A^\dagger rv\right]$. Hence, we get

$$DT(\bar{x}+ru)rv = [I - AA^\dagger]rv - A\left[Df(\bar{x}+ru) - A^\dagger\right]rv, \tag{42}$$

where the first term will be very small. For $k \geq 0$ and $i \in \{1,2\}$, consider the expansion

$$\left(\left[Df(\bar{x}+ru) - A^\dagger\right]rv\right)_{k.i} = \sum_{l_1=1}^{3} c_{k,i}^{(l_1)} r^{l_1}. \tag{43}$$

4.1.1 The bounds $Z_k(r)$, $k \in \{0, \ldots, M-1\}$

For $k \in \{0, \ldots, M-1\}$, we generated the coefficients $c_{k,i}^{(l_1)}$ using *Maple*. Upper bounds $C_k^{(l_1)} \geq 0$ such that $\left|c_{k,i}^{(l_1)}\right| \leq C_k^{(l_1)}$, for $i = 1, 2$ and $k \geq 1$ are presented in Table 1. Note that the cases $c_{0,1}^{(1)}$ and $c_{0,2}^{(1)}$ are treated differently. Indeed, the upper bound $|c_{0,1}^{(1)}| \leq C_{0,1}^{(1)}$ is given in the first line of Table 1 and $c_{0,2}^{(1)} = 0$. Hence, defining

$$C_F^{(l_1)} = \begin{pmatrix} \begin{pmatrix} C_{0,1}^{(1)} \\ 0 \end{pmatrix} \\ \begin{pmatrix} C_k^{(l_1)} \\ C_k^{(l_1)} \end{pmatrix}_{k=1,\ldots,M-1} \end{pmatrix},$$

we get that $|[DT(\bar{x}+ru)rv]_F| \leq_{cw} \left|\left[I_F - A_M \cdot Df^{(M)}(\bar{x})\right] v_F\right| r + \sum_{l_1=1}^{3} |A_M| \cdot C_F^{(l_1)} r^{l_1}$.

The infinite discrete convolution sums in Table 1 can be estimated using the following result.

Lemma 4.2. *Let $k \in \{0, \ldots, M-1\}$, recall the definition of the weights ω_k^s in (8) and define*

$$\phi_k = \sum_{k_1=1}^{k-1} \frac{1}{k_1^s (k-k_1)^s}. \tag{44}$$

$C_{0,1}^{(1)}$	$2 \sum_{k=2m-1}^{\infty} \frac{1}{\omega_k^s}$
$k \in \{0, \ldots, 2m-2\}$.	
$C_k^{(1)}$	$4 \left(\sum_{j=1}^{d} \lambda_j \right) \sum_{k_1=2m-1}^{k+m-1} \frac{\left\lvert \bar{a}_{k-k_1} \right\rvert + \left\lvert \bar{b}_{k-k_1} \right\rvert}{\omega_{k_1}^s}$
$C_k^{(2)}$	$\left(\sum_{j=1}^{d} \lambda_j \tau_j \right) \frac{4k}{\omega_k^s} + \left(\sum_{j=1}^{d} \lambda_j \tau_j^2 \right) (\lvert \bar{a}_k \rvert + \lvert \bar{b}_k \rvert)\, k^2 + \frac{2k}{\omega_k^s} + 8 \left(\sum_{j=1}^{d} \lambda_j \right) \sum_{k_1+k_2=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} + 2 \left(\sum_{j=1}^{d} \lambda_j \tau_j \right) \sum_{k_1=-m+1}^{m-1} \frac{(\lvert k_1 \rvert + \lvert k-k_1 \rvert) \left(\left\lvert \bar{a}_{k_1} \right\rvert + \left\lvert \bar{b}_{k_1} \right\rvert \right)}{\omega_{k-k_1}^s}$ $+2 \left(\sum_{j=1}^{d} \lambda_j \tau_j \right) \sum_{k_1=-m+1+k}^{m-1} \frac{(\lvert k_1 \rvert + \lvert k-k_1 \rvert) \left(\left\lvert \bar{a}_{k_1} \right\rvert + \left\lvert \bar{b}_{k_1} \right\rvert \right)}{\omega_{k-k_1}^s} + \left(\sum_{j=1}^{d} \lambda_j \tau_j^2 \right) \sum_{k_1=-m+1+k}^{m-1} \left(\left\lvert -\bar{a}_{k_1} \bar{a}_{k-k_1} + \bar{b}_{k_1} \bar{b}_{k-k_1} \right\rvert + \left\lvert \bar{a}_{k_1} \bar{b}_{k-k_1} + \bar{b}_{k_1} \bar{a}_{k-k_1} \right\rvert \right) {k_1}^2$
$C_k^{(3)}$	$4 \left(\sum_{j=1}^{d} \lambda_j \tau_j \right) \sum_{k_1+k_2=k} \frac{\lvert k_1 \rvert}{\omega_{k_1}^s \omega_{k_2}^s}$

Table 1: The bounds $C_{k,i}^{(l_1)}$, $i = 1, 2$.

Then

$$\sum_{k_1+k_2=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \leq \phi_k + \frac{1}{\omega_k^s} \left(4 + \frac{2}{s-1} \right) \tag{45}$$

and

$$\sum_{k_1+k_2=k} \frac{|k_1|}{\omega_{k_1}^s \omega_{k_2}^s} \leq \frac{1}{k+1} \left(1 + \frac{1}{2s-3} \right) + \frac{k}{2} \phi_k + \frac{k}{\omega_k^s} + \frac{1}{(k+1)^{s-1}} \left(1 + \frac{1}{s-1} \right). \tag{46}$$

Proof. See [13]. $\square$

The first infinite sum of Table 1 can be taken care with the following.

$$\sum_{k=2m-1}^{\infty} \frac{1}{\omega_k^s} \leq \frac{1}{(s-1)(2m-2)^{s-1}} \tag{47}$$

Hence, replacing the infinite sums in Table 1 using the upper bounds (45), (46) and (47), we get new upper bounds $\mathbf{C}_F^{(l_1)}$. For $k \in \{0, \ldots, M-1\}$, we then define the $Z_k(r) \in \mathbb{R}^2$ as the $k^{th}-$ component of

$$Z_F(r) \stackrel{\text{def}}{=} \left| \left[I_F - A_M \cdot Df^{(M)}(\bar{x}) \right] v_F \right| r + \sum_{l_1=1}^{3} |A_M| \cdot \mathbf{C}_F^{(l_1)} r^{l_1}. \tag{48}$$

4.1.2 The bound $\hat{Z}_M(r)$

For $k \geq M$, we compute upper bounds $\hat{C}^{(l_1)} > 0$ such that for every $k \geq M$ and $i \in \{1, 2\}$,

$$\left| c_{k,i}^{(l_1)} \right| \leq \frac{1}{k^{s-1}} \hat{C}^{(l_1)} \tag{49}$$

where $\hat{C}^{(l_1)}$ is independent of k and i. The derivation of the bounds $\hat{C}^{(l_1)}$ is a combination of symbolic computations using *Maple* and the following result from [13].

Lemma 4.3. *Let $k \geq M = 2m-1$ and define*

$$\gamma \stackrel{\text{def}}{=} 2 \left[\frac{2m-1}{2m-2} \right]^s + \left[\frac{4\ln(2m-3)}{2m-1} + \frac{\pi^2 - 6}{3} \right] \left[\frac{2}{2m-1} + \frac{1}{2} \right]^{s-2},$$

$\widehat{C}^{(1,0)}$	$\left(\sum_{j=1}^{d} \lambda_j\right) \sum_{k_1=1}^{m-1} \frac{4}{2m-1} (\lvert\bar{a}_{k_1}\rvert + \lvert\bar{b}_{k_1}\rvert) \left(1 + \frac{1}{\left(1 - \frac{k_1}{2m-1}\right)^s}\right)$
$\widehat{C}^{(2,0)}$	$2 + 2\left(\sum_{j=1}^{d} \lambda_j \tau_j\right)(1 + \lvert\bar{a}_0\rvert + \lvert 1 + \bar{a}_0\rvert) + \left(\sum_{j=1}^{d} \lambda_j\right) \frac{8}{2m-1}\left(4 + \frac{2}{s-1} + \gamma\right) + \left(\sum_{j=1}^{d} \lambda_j \tau_j\right) \sum_{k_1=1}^{m-1} \frac{2k_1}{2m-1} (\lvert\bar{a}_{k_1}\rvert + \lvert\bar{b}_{k_1}\rvert) \left(1 + \frac{1}{\left(1 - \frac{k_1}{2m-1}\right)^s}\right)$ $+ \left(\sum_{j=1}^{d} \lambda_j \tau_j\right) \sum_{k_1=1}^{m-1} 2(\lvert\bar{a}_{k_1}\rvert + \lvert\bar{b}_{k_1}\rvert) \left(1 + \frac{1}{\left(1 - \frac{k_1}{2m-1}\right)^{s-1}}\right)$
$\widehat{C}^{(3,0)}$	$4\left(\sum_{j=1}^{d} \lambda_j \tau_j\right)\left(3 + \frac{2}{s-1} + \frac{\gamma}{2}\right)$

Table 2: The bounds $\hat{C}^{(l_1)}$.

Then

$$\sum_{k_1+k_2=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \leq \frac{1}{k^{s-1}} \left[\frac{1}{2m-1}\left(4 + \frac{2}{s-1} + \gamma\right)\right] \tag{50}$$

and

$$\sum_{k_1+k_2=k} \frac{|k_1|}{\omega_{k_1}^s \omega_{k_2}^s} \leq \frac{1}{k^{s-1}} \left(3 + \frac{2}{s-1} + \frac{\gamma}{2}\right). \tag{51}$$

The bounds (50) and (51) are used to find the $\hat{C}^{(l_1)}$ satisfying (49). The bounds $\hat{C}^{(l_1)}$ are presented in Table 2.

Lemma 4.4. *Recalling the definition of τ^* in (40) and of ρ given by (28), let*

$$\hat{Z}_M(r) \stackrel{\text{def}}{=} \frac{1}{M^s}\left(\frac{\rho^2\tau^*}{M} + \rho\right)\left[\sum_{l_1=1}^{3} \hat{C}^{(l_1)} r^{l_1}\right]\begin{pmatrix}1\\1\end{pmatrix}. \tag{52}$$

For $k \geq M = 2m-1$,

$$|[DT(\bar{x} + ru)rv]_k| \leq_{cw} \hat{Z}_M(r)\left(\frac{M}{k}\right)^s.$$

Proof. Let $k \geq M$. Combining equations (42) and (43), the proof of Lemma 3.1 and the definition of Ξ in (29),

$$\begin{aligned}\left|[DT(\bar{x}+ru)rv]_k\right| &= \left|-\Lambda_k^{-1}\left[Df(\bar{x}+ru)rv - A^\dagger rv\right]_k\right| \leq_{cw} \sum_{l_1=1}^{3} |\Lambda_k^{-1}| \cdot |c_k^{(l_1)}| r^{l_1} \\ &\leq_{cw} \sum_{l_1=1}^{3} \frac{1}{k}\Xi \cdot \frac{1}{k^{s-1}}\hat{C}^{(l_1)}\begin{pmatrix}1\\1\end{pmatrix} r^{l_1} = \hat{Z}_M(r)\left(\frac{M}{k}\right)^s.\end{aligned}$$

□

4.1.3 Definition of the radii polynomials

We have now all the ingredients to define the radii polynomials for the class of problem (38). Recalling the bound Y_F given by (37), the bound $Z_F(r)$ given by (48) and the bound $\hat{Z}_M(r)$ given by (52), we define the radii polynomials $p_k \in \mathbb{R}^2$ by

$$p_k(r) \stackrel{\text{def}}{=} \begin{cases} |A_M f_F(\bar{x})|_k + \left|\left[I_F - A_M \cdot Df^{(M)}(\bar{x})\right] v_F\right|_k r \\ +\sum_{l_1=1}^{3} \left(|A_M| \cdot \mathbf{C}_F^{(l_1)}\right)_k r^{l_1} - \frac{r}{\omega_k^s}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), & k = 0, \ldots, M-1; \\ \left(\frac{1}{M^s}\left(\frac{\rho^2\tau^*}{M} + \rho\right)\sum_{l_1=1}^{3} \hat{C}^{(l_1)} r^{l_1} - \frac{r}{\omega_M^s}\right)\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), & k = M. \end{cases} \tag{53}$$

4.2 Proofs of Theorem 1.2 and Theorem 1.3

The proofs of Theorem 1.2 and Theorem 1.3 is based on the success of the following procedure.

Procedure 4.5. *To check the hypotheses of Lemma 3.4 in the context of proving the existence of periodic solutions of (38), we proceed as follows.*

1. *Fix d time lags $\tau_1, \ldots, \tau_d$ and real numbers $\lambda_1, \ldots, \lambda_d$. Fix a decay rate s, a finite reduction dimension m, $M = 2m-1$, and an approximate solution $\hat{x}_F$ of $f^{(m)}(\cdot) = 0$.*

2. *With a Newton iteration, find near $\hat{x}_F$ an approximate solution $\bar{x}_F$ of $f^{(m)}(x_F) = 0$. Compute exactly the derivative $D_x f^{(M)}(\bar{x}_F)$ and compute an approximate inverse A_M of the inverse of $D_x f^{(M)}(\bar{x}_F)$. Using interval arithmetic, verify that conditions (24) and (41) are satisfied (this guarantees that the linear operator A defined in (27) is invertible).*

3. *Compute, using interval arithmetic, the coefficients of Table 1 and Table 2 to complete the construction of the radii polynomials p_k, $k = 0, \ldots, M$ given by equation (53).*

4. *Calculate numerically $\mathcal{I} = [r_1^-, r_1^+] \stackrel{\text{def}}{=} \bigcap_{k=0}^{M} \{r \geq 0 \,|\, p_k(r) \leq 0\}$.*

5.
   - *If $\mathcal{I} = \varnothing$ then go to Step 7.*
   - *If $\mathcal{I} \neq \varnothing$ then let $r = \frac{r_1^- + r_1^+}{2}$. Compute with interval arithmetic $p_k(r)$. If $p_k(r) < 0$ for all $k = 0, \ldots, M$ then go to Step 6; else go to Step 7.*

6. *The proof has succeeded. By Lemma 3.4, the operator T given by (30) has a unique fixed point*

$$\tilde{x} = \left(\tilde{L}, \tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \tilde{a}_3, \tilde{b}_3, \ldots\right) \in \Omega^s$$

   within the ball $B_{\bar{x}}(r)$. By the invertibility of the linear operator A given by (27) and by the equivalence given in Lemma 2.4, one gets the existence of a periodic solution

$$\tilde{y}(t) = \tilde{a}_0 + 2\sum_{k=1}^{\infty}\left[\tilde{a}_k \cos k\tilde{L}t - \tilde{b}_k \sin k\tilde{L}t\right]$$

   such that $\tilde{y}(0) = \psi_0 = 0$.

7. *The proof has failed. Either increase $M > 2m-1$ or increase m and return to Step 2.*

4.2.1 Proof of Theorem 1.2

Fix $\tau_1 = 1.65$, $\tau_2 = 0.35$ and consider λ as a free parameter in

$$y'(t) = -\lambda \left[y(t-\tau_1) + y(t-\tau_2)\right] \left[1 + y(t)\right]. \tag{54}$$

Hence, the problem $f = 0$ with f given by (17) can be considered as being a parameter dependent problem of the form

$$f(x, \lambda) = 0. \tag{55}$$

Let $m_1 = 60$, $m_2 = 80$ and $m_3 = 78$. Using a pseudo arc-length continuation algorithm (e.g. see [23]) on a finite dimensional Galerkin projection of (55), one obtained three numerical approximations $\bar{x}^{(i)} = \bar{x}_F^{(i)} \in \mathbb{R}^{2m_i}$ $(i = 1, 2, 3)$ of $f^{(m_i)}(\cdot) = 0$ at the parameter value $\lambda = 3.4$. Consider the decay rates $s_1 = 3$, $s_2 = 2.5$ and $s_3 = 2$.

For each $i = 1, 2, 3$, we applied Procedure 4.5 with $m = m_i$, $M = 2m_i - 1$, $s = s_i$ to get the existence of $r_i > 0$ such that $B_i \stackrel{\text{def}}{=} B_{\bar{x}^{(i)}}(r_i)$ contained a unique zero of (55) at the parameter value $\lambda = 3.4$. We showed that the sets B_1, B_2, B_3 were mutually disjoint. Hence, to the three distinct solutions $\tilde{x}_i \in B_i$ $(i = 1, 2, 3)$ correspond three distinct periodic solutions $\tilde{y}_i(t)$ of (54) such that $\tilde{y}_i(0) = 0$. The computer program *proof_gen_wright3d.m* in [27] using the interval arithmetic package *Intlab* (see [24]) developed for the software *Matlab* performed successfully the above steps. That completes the proof of Theorem 1.2.

4.2.2 Proof of Theorem 1.3

Fix $\lambda_1 = \lambda_2 = 2.425$, $\tau_1 = 1.65$, $\tau_2 = 0.35$, $\tau_3 = 1$ and consider λ_3 as a free parameter in

$$y'(t) = -\left[\lambda_1 y(t-\tau_1) + \lambda_2 y(t-\tau_2) + \lambda_3 y(t-\tau_3)\right] \left[1 + y(t)\right]. \tag{56}$$

Hence, the problem $f = 0$ with f given by (17) can be considered as being a parameter dependent problem of the form

$$f(x, \lambda_3) = 0. \tag{57}$$

Let $m = 40$ and for each $k = 0, \ldots, 25$, let $\lambda_3^{(k)} \stackrel{\text{def}}{=} \frac{k}{100}$. Using a continuation algorithm on a finite dimensional Galerkin projection of (57), one obtained, for each $k = 0, \ldots, 25$, a numerical approximations $\bar{x}^{(k)} = \bar{x}_F^{(k)} \in \mathbb{R}^{2m}$ of $f^{(m)}(\cdot, \lambda_3^{(k)}) = 0$.

For each $k = 0, \ldots, 25$, we applied Procedure 4.5 with $M = 2m - 1$, $\bar{x}_F = \bar{x}_F^{(k)}$ and $s = 3$ to get the existence of $r_k > 0$ such that $B_k \stackrel{\text{def}}{=} B_{\bar{x}^{(k)}}(r_k)$ contained a unique zero of (57) at the parameter value $\lambda_3^{(k)}$. Hence, for each $k = 0, \ldots 25$, one has the existence of a solution $\tilde{x}_k \in B_k$ which corresponds to a nontrivial periodic solution $\tilde{y}_k(t)$ of (56) such that $\tilde{y}_k(0) = 0$. Note that the numerical data associated to the case $k = 25$ can be found in Figure 5. As for the proof of Theorem 1.2, the computer program *proof_gen_wright3d.m* that can be found in [27] performed successfully the above steps. That completes the proof of Theorem 1.3.

5 Future Projects

A first future project considers the possibility of extending our new proposed computational method to study the existence of invariant tori in delay equations with multiple delays. Let us be more specific about that. We would like to study the effect of the presence of an instantaneous term in (2) by considering the equation

$$y' = -\eta y(t) - \lambda\left[y(t-\tau_1) + y(t-\tau_2)\right]\left[1 + y(t)\right] \tag{58}$$

where $\eta \in \mathbb{R}$. Notice that the substitution $\eta = 0$ recovers (2). Looking for the boundary of the stability region of the trivial solution $x \equiv 0$ of (58), one needs to compute the so–called *Hopf–curves*, curves in the parameter space.

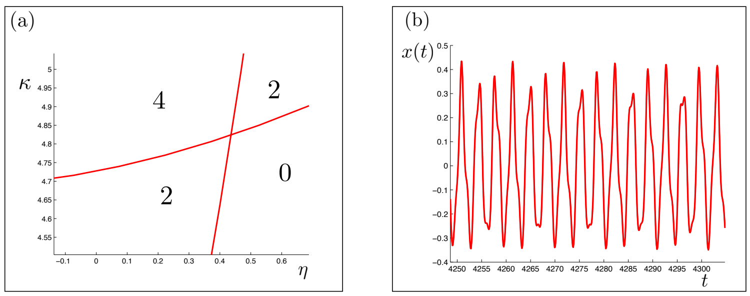

Figure 3: (a) Hopf curves as boundary of the stability region of the linearization of (58) where $\kappa = 2\lambda$. The numbers indicate the number of unstable roots in each region. (b) Solution profile of the solution of equation (58) when $\eta = 0.4$ $\kappa = 4.85$.

Using the ideas of [25, 26], one can plot these curves and count the number of the unstable roots corresponding to the trivial solution of (58). In Figure 3(a) we plot a close up on the parameter space with the Hopf–curves of (58). As one can observe in Figure 3, equation (58) can undergo a more complicated bifurcation then its counterpart with one delay. More precisely, one can expect a quasi–periodic solution as an outcome of a double Hopf bifurcation. As Figure 3(b) suggests, this expectation is reasonable since irregular oscillations around zero can be observed in the solution profile. With the aid of a shifted coordinate system we can embed the time series into the two–dimensional Euclidean space.

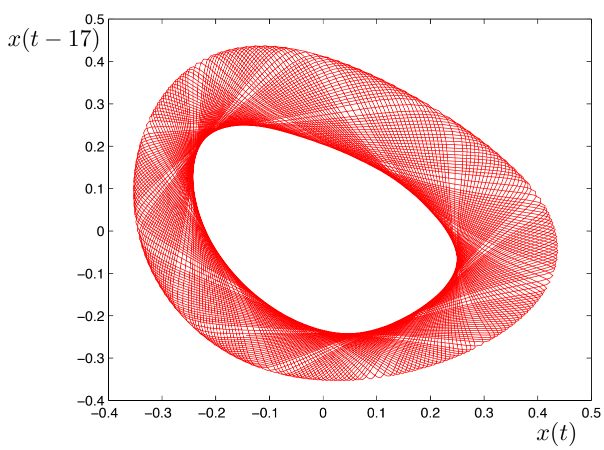

Figure 4: Numerically observed invariant torus. The two–dimensional embedding of the time series plotted in Figure (3)

The plot strengthen our argument about the possibility of the existence of a quasi–periodic solution for (58) or an invariant torus in the state–space. The idea of this project would then be to expand the numerically observed invariant torus using Fourier series and to try to prove its existence using a contraction mapping argument. This is a possibility which is currently under investigation.

A second future project would be to introduce a different set up for the nonlinear operator f whose zeros corresponds to periodic solutions of (1). More precisely, in order to enlarge the class of equations (1) to problems with more general nonlinearities (i.e. not only polynomial nonlinearities), one would like to represent the periodic solutions using a different approximation than Fourier series. We believe that splines (i.e. piecewise polynomials) may be a natural choice for this new approximation. Ideas similar to the ones introduced in [22] for the study of connecting orbits may be helpful in this context. This is work in progress.

A last interesting future project would be to extend our computational fixed point method to other types of functional differential equations, namely neutral equations and equations of mixed type.

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Appendix: Data for a periodic solution of Theorem 1.3

L	1.769416601644631	$\bar{a}_0$	-0.000000000000018
$\bar{a}_1$	0.601786339573172	$\bar{b}_1$	0.504704483605188
$\bar{a}_2$	0.019808638462510	$\bar{b}_2$	0.551034215919532
$\bar{a}_3$	-0.204621598835984	$\bar{b}_3$	0.333257682739841
$\bar{a}_4$	-0.197074262548811	$\bar{b}_4$	0.109965255095155
$\bar{a}_5$	-0.132090298002992	$\bar{b}_5$	-0.003914262849352
$\bar{a}_6$	-0.065741598794274	$\bar{b}_6$	-0.031920282037413
$\bar{a}_7$	-0.022758737562314	$\bar{b}_7$	-0.030666374740040
$\bar{a}_8$	-0.005809569627743	$\bar{b}_8$	-0.021164734340616
$\bar{a}_9$	-0.000062041227429	$\bar{b}_9$	-0.010921573163981
$\bar{a}_{10}$	0.002103033279485	$\bar{b}_{10}$	-0.005032281663922
$\bar{a}_{11}$	0.001938791853807	$\bar{b}_{11}$	-0.002333672359275
$\bar{a}_{12}$	0.001181909800654	$\bar{b}_{12}$	-0.000833836698067
$\bar{a}_{13}$	0.000695681613703	$\bar{b}_{13}$	-0.000181135543301
$\bar{a}_{14}$	0.000363862567645	$\bar{b}_{14}$	-0.000013208309489
$\bar{a}_{15}$	0.000159861579587	$\bar{b}_{15}$	0.000034705938832
$\bar{a}_{16}$	0.000072149838004	$\bar{b}_{16}$	0.000039242056107
$\bar{a}_{17}$	0.000031374518281	$\bar{b}_{17}$	0.000024120035334
$\bar{a}_{18}$	0.000011188419668	$\bar{b}_{18}$	0.000013747409265
$\bar{a}_{19}$	0.000003979829238	$\bar{b}_{19}$	0.000007576004123
$\bar{a}_{20}$	0.000001236364323	$\bar{b}_{20}$	0.000003557840018
$\bar{a}_{21}$	0.000000199149924	$\bar{b}_{21}$	0.000001722109911
$\bar{a}_{22}$	0.000000022651239	$\bar{b}_{22}$	0.000000832013628
$\bar{a}_{23}$	-0.000000038533044	$\bar{b}_{23}$	0.000000352794201
$\bar{a}_{24}$	-0.000000056869626	$\bar{b}_{24}$	0.000000165344387
$\bar{a}_{25}$	-0.000000030780276	$\bar{b}_{25}$	0.000000078365641
$\bar{a}_{26}$	-0.000000016762204	$\bar{b}_{26}$	0.000000030763716
$\bar{a}_{27}$	-0.000000010418774	$\bar{b}_{27}$	0.000000013383477
$\bar{a}_{28}$	-0.000000004908838	$\bar{b}_{28}$	0.000000006103783
$\bar{a}_{29}$	-0.000000002357079	$\bar{b}_{29}$	0.000000002309899
$\bar{a}_{30}$	-0.000000001214979	$\bar{b}_{30}$	0.000000000899929
$\bar{a}_{31}$	-0.000000000564721	$\bar{b}_{31}$	0.000000000336531
$\bar{a}_{32}$	-0.000000000269476	$\bar{b}_{32}$	0.000000000117723
$\bar{a}_{33}$	-0.000000000119565	$\bar{b}_{33}$	0.000000000049303
$\bar{a}_{34}$	-0.000000000052694	$\bar{b}_{34}$	0.000000000014557
$\bar{a}_{35}$	-0.000000000028687	$\bar{b}_{35}$	0.000000000005479
$\bar{a}_{36}$	-0.000000000012958	$\bar{b}_{36}$	0.000000000004276
$\bar{a}_{37}$	-0.000000000005090	$\bar{b}_{37}$	0.000000000001491
$\bar{a}_{38}$	-0.000000000002523	$\bar{b}_{38}$	0.000000000000523
$\bar{a}_{39}$	-0.000000000001150	$\bar{b}_{39}$	0.000000000000399

Figure 5: Coordinates of the center $\bar{x}$ of the set $B_{\lambda_3} \subset \Omega^3$ from Theorem 1.3 at the last parameter value $\lambda_3 = \frac{1}{4}$. More explicitly $B_{\lambda_3} = B_{\bar{x}}(r) = \bar{x} + B(r)$, where $r = 3.5 \times 10^{-7}$ and $B(r)$ is given by (31). Since $s = 3$, by equation (35) in Lemma 3.2, the ℓ^2–error between $\bar{x}$ and the unique solution $\tilde{x} \in B_{\lambda_3}$ satisfies $\|\tilde{x} - \bar{x}\|_{\ell^2} \leq 7.35 \times 10^{-7}$.