Rational Points and Hypergeometric Functions

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The Goal

Let X_λ be the family of varieties defined by

$$X_\lambda : x_1^d + \cdots + x_n^d - d\lambda x_1^{h_1} \cdots x_n^{h_n} = 0$$

where each h_i is a positive integer, $\sum h_i = d$ and $\gcd(d, h_1, \ldots, h_n) = 1$ and $\lambda \in \mathbb{F}_q$.

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Objective: Find an explicit relation between the function $N_{\mathbb{F}_q}(\lambda)$ and hypergeometric functions.

Motivation - Igusa's result

It is known that for the Legendre family of elliptic curves:

$$E_\lambda : y^2 = x(x-1)(x-\lambda),$$

we get that

$$N_{\mathbb{F}_p}(\lambda) \equiv (-1)^{\frac{p-1}{2}} \left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| \lambda \right) \right]_0^{\frac{p-1}{2}} \bmod p.$$

We also know that ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1|\lambda)$ is the only holomorphic solution around 0 of the Picard-Fuchs differential equation satisfied by the periods of E_λ.

The generalized hypergeometric function

Let $A, B \in \mathbb{N}$. A **hypergeometric function** is a function on $\mathbb{C}$ of the form:

$$
\begin{aligned}
{}_AF_B(\alpha;\beta|z) &= {}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B|z) \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_A)_k}{(\beta_1)_k\cdots(\beta_B)_k k!} z^k,
\end{aligned}
$$

where $\alpha \in \mathbb{Q}^A$ are *numerator parameters*, $\beta \in \mathbb{Q}^B$ are *denominator parameters*, and the Pochhammer notation is defined by:

$$(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Gauss sums

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- Let $\psi : \mathbb{F}_q \to K^*$ be a (fixed) additive character.
- For $s \in \frac{1}{(q-1)}\mathbb{Z}/\mathbb{Z}$ we let $g(s)$ denote the Gauss sum

$$g(s) = \sum_{x \in \mathbb{F}_q} \chi_s(x)\psi(x)$$

A large group action

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The varieties X_λ allow a faithful action of the group

$$G = \{\xi \in \mu_d^n | \xi^h = 1\}/\Delta,$$

by $\xi = (\xi_1, \ldots, \xi_n)$ taking the point $(x_1, \ldots, x_n)$ to $(\xi_1 x_1, \ldots, \xi_n x_n)$.

A large group action

$$char(G) \leftrightarrow W,$$

where

$$W = \{(w_1, \dots, w_n) | 0 \leq w_i < d, \sum w_i \equiv 0 \mod d\},$$

and $w' \sim w$ if $w - w'$ is a multiple (mod d) of h.
Here

$$\chi_w(\xi) := \chi(\xi^w), \qquad \xi^w = \xi_1^{w_1} \cdots \xi_n^{w_n}$$

and χ is a fixed primitive character of μ_d, which we can get for example by restricting $\chi_{1/(q-1)}$ to μ_d.

Koblitz's result

Assume $d|q-1$.

Theorem (Koblitz)

$$N_{\mathbb{F}_q}(\lambda) = N_{\mathbb{F}_q}(0) + \frac{1}{q-1} \sum_{\substack{s \in \frac{d}{q-1}\mathbb{Z}/\mathbb{Z} \\ w \in W}} \frac{g\left(\frac{w+sh}{d}\right)}{g(s)} \chi_s(d\lambda),$$

where we denote $g\left(\frac{w+sh}{d}\right) = \prod_i g\left(\frac{w_i+sh_i}{d}\right)$.

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Theorem (Gross-Koblitz)

For $s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}$, we have

$$g(s) = -(-p)^s \Gamma_p(s).$$

Here, Γ_p is the p-adic analog of the Gamma function.

The 0-dimensional family

Study $N_{\mathbb{F}_p}(\lambda)$ mod p for the family

$$Z_\lambda : x_1^d + x_2^d - d\lambda x_1 x_2^{d-1} = 0.$$

Assume p is a prime such that $d|p-1$.

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Formula (S)

$$N_{\mathbb{F}_p}(\lambda) = N_{\mathbb{F}_p}(0) + \frac{-1}{p-1}\sum_{a=0}^{p-2} \frac{(-p)^{\eta(a)}\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\left\{\frac{(d-1)a}{p-1}\right\}\right)}{\Gamma_p\left(\left\{\frac{da}{p-1}\right\}\right)}\omega(d\lambda)^{-da}$$

where $\eta(a) = \left(\frac{a}{p-1} + \{\frac{(d-1)a}{p-1}\} - \{\frac{da}{p-1}\}\right)$.

Notation

- $\omega : \mathbb{F}_p^* \to \mathbb{C}_p^*$ - Teichmüller character. ($\omega(x) \equiv x \bmod p$)
- $\{x\} = x - [x]$, fractional part of x.

The 0-dimensional family

Theorem (S)

Let $\alpha^{(0)} = (\frac{1}{d}, \ldots, \frac{d-1}{d}), \beta^{(0)} = (\frac{1}{d-1}, \ldots, \frac{d-2}{d-1})$.

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0)$$
$$\equiv \sum_{i=0}^{d-2} \left[{}_dF_{d-1}(\alpha^{(i)}; \beta^{(i)} | (d-1)^{-(d-1)}\lambda^{-d}) \right]_{\frac{i(p-1)}{d-1}}^{\frac{(i+1)(p-1)}{d}-1} \bmod p,$$

where $\alpha^{(i)} = \left(\frac{1}{d}+1, \ldots, \frac{i}{d}+1, \frac{i+1}{d}, \ldots, \frac{d-1}{d}\right)$, *and*
$\beta^{(i)} = \left(\frac{1}{d-1}+1, \ldots, \frac{i}{d-1}+1, \frac{i+1}{d-1}, \ldots, \frac{d-2}{d-1}\right)$.

$[u(z)]_i^j$ denotes the polynomial which is the truncation of a series $u(z)$ from $n = i$ to j.

The 0-dimensional family

So for example in the case $d = 3$ we get that

$$\begin{aligned} N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) &\equiv \left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2} \middle| \frac{1}{2^2\lambda^3} \right) \right]_0^{\frac{p-1}{3}-1} \\ &+ \left[{}_2F_1\left(\frac{4}{3}, \frac{2}{3}; \frac{3}{2} \middle| \frac{1}{2^2\lambda^3} \right) \right]_{\frac{p-1}{2}}^{\frac{2(p-1)}{3}-1} \mod p. \end{aligned}$$

The Dwork family with $d = 4$

$$Y_\lambda : x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\lambda x_1 x_2 x_3 x_4 = 0.$$

The set W is made up of 64 vectors, but we can split them up into 16 equivalence classes, and of those there are only three "types". These are

$$(0,0,0,0),(1,1,1,1),(2,2,2,2),(3,3,3,3)$$

$$(0,1,1,2),(1,2,2,3),(2,3,3,0),(3,0,0,1)$$

$$(0,0,2,2),(1,1,3,3),(2,2,0,0),(3,3,1,1)$$

The rest are permutations of these. So there is one class of the first type, 12 classes of the second type, and 3 classes of the third type.

The Dwork family with $d = 4$

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^4}{g(4s)} \chi_{4s}(4\lambda) \qquad (S_1)$$

$$+\frac{12}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)g(s+\frac{1}{4})^2 g(s+\frac{1}{2})}{g(4s)} \chi_{4s}(4\lambda) \qquad (S_2)$$

$$+\frac{3}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^2 g(s+\frac{1}{2})^2}{g(4s)} \chi_{4s}(4\lambda). \qquad (S_3)$$

The Dwork family with $d = 4$

Using Gross-Koblitz and taking mod p leaves only (S_1), so

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \equiv \left[{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1 \,\middle|\, \lambda^{-4} \right) \right]_0^{\frac{p-1}{4}-1} \mod p$$

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- Extend the results for $\mathbb{F}_p$ points to $\mathbb{F}_q$.

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- Relate the number of points to eigenvalues of Frobenius.

Thanks!

Thanks for the invitation!