A newform theory for Hilbert Eisenstein series

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Eisenstein series of weight $k \geq 3$

Let $M_k(N, \psi)$ denote the space of modular forms of weight k, level N and character ψ and $E_k(N, \psi)$ the subspace of Eisenstein series.

Let ψ_1 and ψ_2 be Dirichlet characters modulo N_1 and N_2.

Associated to these characters is an Eisenstein series $E_{\psi_1,\psi_2}(z) = \sum_{n=0}^{\infty} a(n) q^n \in E_k(N_1 N_2, \psi_1 \psi_2)$ where $q = e^{2\pi i z}$ and

$$a(n) = \begin{cases} -L(0, \psi_1) L(1-k, \psi_2) & \text{if } n = 0 \\ \sum_{d|n} \psi_1(n/d) \psi_2(d) d^{k-1} & \text{if } n > 0. \end{cases}$$

For primes $p \nmid N_1 N_2$, one sees that $E_{\psi_1,\psi_2} \mid T_p = a(p) E_{\psi_1,\psi_2}$.

We say that E_{ψ_1,ψ_2} is a **newform** if ψ_1 and ψ_2 are primitive.

Note that the exact level of a newform is always $\mathrm{cond}(\psi_1)\mathrm{cond}(\psi_2)$.

Generalizing Atkin-Lehner theory, Weisinger proved that $E_k(N,\psi)$ has a basis of newforms of level M dividing N and their shifts by the divisors of NM^{-1}.

Weisinger also proved a multiplicity-one theorem showing that, in analogy with cuspidal newforms, Eisenstein newforms are uniquely determined by their Hecke eigenvalues.

Theorem (Weisinger): *Let* $f = \sum a(n)q^n$ *and* $g = \sum b(n)q^n$ *be Eisenstein newforms of character* ψ *and levels* N *and* M *respectively. If* $a(p) = b(p)$ *for all but finitely many primes* p *then* $N = M$ *and* $f = g$.

Let F be a totally real number field, $\mathcal{N}$ be an integral ideal of $\mathcal{O}_F$ and $\mathcal{E}_k(\mathcal{N},\Psi)$ be the space of Hilbert modular Eisenstein series over F of parallel weight $k \geq 3$, level $\mathcal{N}$ and Hecke character Ψ.

Although Eisenstein series $f \in \mathcal{E}_k(\mathcal{N},\Psi)$ do not in general have Fourier expansions when the class number of F is greater than 1, one may still associate to f a set of 'Fourier coefficients' $a(\mathfrak{m}, f)$ indexed by integral ideals $\mathfrak{m}$ of $\mathcal{O}_F$. The coefficients $a(\mathfrak{m}, f)$ uniquely determine f.

Let ψ_1 (resp. ψ_2) be a character on the strict ideal class group of F modulo $\mathcal{N}_1$ (resp. $\mathcal{N}_2$).

Let Ψ be the Hecke character such that $\Psi^*(\mathfrak{r}) = (\psi_1\psi_2)(\mathfrak{r})$ for $\mathfrak{r}$ coprime to $\mathcal{N}_1\mathcal{N}_2$.

It is a theorem of Shimura that there exists an Eisenstein series $E_{\psi_1,\psi_2} \in \mathcal{E}_k(\mathcal{N}_1\mathcal{N}_2, \Psi)$ such that

$$a(\mathfrak{m}, E_{\psi_1,\psi_2}) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi_1(\mathfrak{m}\mathfrak{r}^{-1})\psi_2(\mathfrak{r})N(\mathfrak{r})^{k-1}.$$

Formulae for the constant terms of these Eisenstein series have recently been computed by Darmon, Dasgupta and Pollack.

For primes $\mathfrak{p} \nmid \mathcal{N}_1\mathcal{N}_2$, one sees that $E_{\psi_1,\psi_2} \mid T_{\mathfrak{p}} = a(\mathfrak{p}, E_{\psi_1,\psi_2})E_{\psi_1,\psi_2}$.

We say that E_{ψ_1,ψ_2} is a **newform** if ψ_1 and ψ_2 are primitive.

Wiles generalized Weisinger's theorem by showing that $\mathcal{E}_k(\mathcal{N},\Psi)$ has a basis of newforms of level $\mathcal{M}$ dividing $\mathcal{N}$ and their shifts by the divisors of $\mathcal{N}\mathcal{M}^{-1}$.

As was the case with elliptic modular Eisenstein series, Hilbert Eisenstein newforms are uniquely determined by their Hecke eigenvalues.

Theorem (Atwill, L.): *Let* $f \in \mathcal{E}_k(\mathcal{N}, \Psi)$ *and* $g \in \mathcal{E}_k(\mathcal{M}, \Psi)$ *be newforms and suppose that* $a(\mathfrak{p}, f) = a(\mathfrak{p}, g)$ *for any set of primes with Dirichlet density strictly greater than* $1/2$*. Then* $\mathcal{M} = \mathcal{N}$ *and* $f = g$*.*

1. This theorem is best possible in the sense that one may construct distinct newforms whose Hecke eigenvalues agree for a set of primes having Dirichlet density equal to $1/2$.

2. This complements Ramakrishnan's refinement of the strong multiplicity-one theorem for cuspidal newforms which shows that cuspidal newforms are uniquely determined by their Hecke eigenforms for any set of primes having Dirichlet density greater than $7/8$.

Application: Diagonalizing the space $\mathcal{M}_k(\mathcal{N}, \Psi)$

Let $\mathcal{M}_k(\mathcal{N}, \Psi)$ denote the space of Hilbert modular forms over F of parallel weight $k \geq 3$, level $\mathcal{N}$ and Hecke character Ψ, and $\mathcal{S}_k(\mathcal{N}, \Psi)$ the subspace of cusp forms.

Pizer introduced, for $F = \mathbb{Q}$ and $\Psi = 1$, an operator $C_{\mathfrak{q}}$ (for $\mathfrak{q} \mid \mathcal{N}$) on $\mathcal{S}_k(\mathcal{N}, \Psi)$ whose action on the subspace generated by newforms of level $\mathcal{N}$ coincides with the action of $T_{\mathfrak{q}}$ and such that $\mathcal{S}_k(\mathcal{N}, \Psi)$ can be decomposed into a direct sum of common eigenspaces, each of dimension 1, for the algebra generated by the operators $T_{\mathfrak{p}}$ (for $\mathfrak{p} \nmid \mathcal{N}$) and $C_{\mathfrak{q}}$ (for $\mathfrak{q} \mid \mathcal{N}$).

Pizer's work was generalized to arbitrary Ψ by Li.

This was later extended by Atwill to allow for arbitrary F as well.

As an application of the newform theory we developed for Hilbert modular Eisenstein series, we are able to prove

Theorem (Atwill, L.): *The space $\mathcal{E}_k(\mathcal{N}, \Psi)$ may be decomposed into a direct sum of common eigenspaces, each of dimension one, for the algebra generated by the Hecke operators $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$* and the operators $\{C_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$.

By using standard estimates on the growth of Hecke eigenvalues we obtain the following corollary.

Corollary: $\mathcal{M}_k(\mathcal{N}, \Psi)$ *may be decomposed into a direct sum of common eigenspaces, each of dimension one, for the algebra generated by the Hecke operators $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$* and the operators $\{C_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$.

Application: Obstruction Sets

Our final application uses the newform theory developed for $\mathcal{E}_k(\mathcal{N}, \Psi)$ to study Eisenstein series which are eigenforms for some, but not all, of the Hecke operators $T_\mathfrak{p}$.

Such forms arise naturally in the theory of quadratic forms; for instance, as the **genus theta series** of a positive definite quadratic form.

Let $f \in \mathcal{E}_k(\mathcal{N}, \Psi)$ and $S(f)$ be the set of finite primes of F for which f is not a $T_\mathfrak{p}$-eigenform. We call $S(f)$ the **obstruction set** of f.

It is clear that $S(f)$ is empty if and only if f is a simultaneous eigenform for all of the Hecke operators $T_\mathfrak{p}$.

Let $\delta(S(f))$ denote the Dirichlet density of $S(f)$ relative to the set of finite primes of F.

Proposition: *Let m be the number of strict ideal classes modulo $\mathcal{N}$. Then $\delta(S(f))$ is an integer multiple of $1/m$.*

It turns out that not all integer multiples of $1/m$ are attained in this manner.

Proposition: *If $\delta(S(f)) < 1/2$ then f is a simultaneous eigenform for all $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathcal{N}$, and $\delta(S(f)) = 0$.*

Remark: The last proposition makes essential use of the orthogonality relations for characters on finite abelian groups.

We have seen that every $f \in \mathcal{E}_k(\mathcal{N}, \Psi)$ can be written uniquely as a linear combination of newforms of level $\mathcal{M}$ dividing $\mathcal{N}$ and their shifts by divisors of $\mathcal{N}\mathcal{M}^{-1}$.

Define f to be a sum of t **classes** of newforms if shifts of exactly t distinct newforms appear in the aforementioned decomposition.

Theorem (Atwill, L.): *Let m be the number of strict ideal classes modulo $\mathcal{N}$ and assume that $m > 1$. Write $\delta(S(f)) = x/m$ for some integer $0 \leq x < m$. Then f is a sum of at most $\lfloor \frac{m}{m-x} \rfloor$ classes of newforms.*

An immediate consequence of the last theorem is that if $\delta(S(f)) = 1/2$ then f is a linear combination of 2 classes of newforms.

Many examples of Eisenstein series f satisfying $\delta(S(f)) = 1/2$ are known. For example, Kitaoka has shown that the genus theta series of any positive definite quadratic form of prime level satisfies this condition.