Realizing certain p-groups as Galois groups

Andrew Schultz

Wellesley College

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- Find L/K so that $\mathrm{Gal}(L/K) \simeq N$
- “Stitching” condition: does Galois S.E.S. match?

A working example: the Heisenberg group

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- One of the two nonabelian groups of order p^3
- Realized by $\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\} \subseteq GL(\mathbb{F}_p)$

Heisenberg via embeddings: classic approach

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If $\xi_p \in F$, then $K = F(\sqrt[p]{a}, \sqrt[p]{b})$ and $L = K(\sqrt[p]{z})$

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Stitching condition

$\exists x \in F(\sqrt[p]{a})$ with $N_{F(\sqrt[p]{a})/F} x = b$, and

$$z = r\ x^{p-1}\sigma\left(x^{p-2}\right)\cdots\sigma^{p-2}(x) \text{ for some } r \in F^\times$$

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Fact: L/F Galois iff N is $\mathbb{F}_p[G]$-module

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Fact: L/F Galois iff N is $\mathbb{F}_p[G]$-module

Fact: M is indecomposable and $\dim_{\mathbb{F}_p}(M) = \ell$ implies

$$M \simeq A_\ell := \mathbb{F}_p[G]/(\sigma - 1)^\ell$$

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$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \longrightarrow H_{p^3} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1$$

$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} = \mathrm{Gal}(L/K), \qquad \mathbb{Z}/p\mathbb{Z} = \mathrm{Gal}(K/F)$$

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- $N \simeq A_1^{\oplus 2}$ $\longleftarrow$ this makes $\mathrm{Gal}(L/F)$ abelian
- $N \simeq A_2$

Bad news: $N \simeq A_2$ can't be enough to ensure $\mathrm{Gal}(L/F) \simeq H_{p^3}$

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- $\gamma \in \ker(e) \implies \mathrm{Gal}(L/F) \simeq A_\ell \rtimes G$
- $\gamma \notin \ker(e) \implies \mathrm{Gal}(L/F) \simeq A_\ell \bullet G$

Finishing our example

$$\left\{\begin{array}{c} G \simeq \mathbb{Z}/p \\ \langle\gamma\rangle \simeq A_2 \\ \gamma \in \ker(e) \end{array}\right\}$$

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$$M_{p^3} \simeq A_2 \bullet G \qquad H_{p^3} \simeq A_2 \rtimes G \qquad (\mathbb{Z}/p)^{\oplus 3} \qquad \mathbb{Z}/p \oplus \mathbb{Z}/p^2$$

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Replace $K^\times/K^{\times p}$ with appropriate parameterizing space

$$J(K) = \begin{cases} K^\times/K^{\times p} & \text{if } \xi_p \in F \\ K/\wp(K) & \text{if } \mathrm{char}(F) = p \\ K(\xi_p)^\times/K(\xi_p)^{\times p}|_{\epsilon=t} & \text{if } \mathrm{char}(F) \neq p \text{ and } \xi_p \notin F \end{cases}$$

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Moral

Solvability of particular embedding problem determined by existence of certain modules in $J(K)$

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Solvability of embedding problems determined by existence of appropriate modules in $J(K)$. And we know structure of $J(K)$!

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$$J(K) \simeq \langle \chi \rangle \oplus Y_0 \oplus \cdots \oplus Y_n$$

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- $Y_i \simeq \left(A_{p^i}\right)^{\oplus \mathfrak{d}_i}$ and $\langle \chi \rangle \simeq A_{p^{i(K/F)}+1}$

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where

- $Y_i \simeq \left(A_{p^i}\right)^{\oplus \mathfrak{d}_i}$ and $\langle \chi \rangle \simeq A_{p^{i(K/F)}+1}$
- $\chi \notin \ker(e)$ and $Y_i \subseteq \ker(e)$ for $0 \leq i < n$,

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- Enumerate extensions with prescribed Galois groups
- Put structural restrictions on absolute Galois groups

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Example

$\nu(\mathbb{Z}/n) = 1$, since $\mathbb{F}_p$ has only one $\mathbb{Z}/n$ extension

Relating M_{p^3} and H_{p^3}

Brattstrom proved: if $\xi_{p^2} \in F$ or $\text{char}(F) = p$, then

$$\nu(M_{p^3}, F) = (p^2 - 1)\nu(H_{p^3}, F)$$

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Using modules, we can prove

$$\nu(M_{p^3}, F) = (p^2-1)\nu(H_{p^3}, F) + \left(\binom{\dim J(F)}{1}_p - \binom{\dim \mathfrak{N}}{1}_p \right) \frac{|J(F)|}{p^2}$$

where $\mathfrak{N}$ is subspace of $J(F)$ where $\mathbb{Z}/p^2 \twoheadrightarrow \mathbb{Z}/p$ is solvable

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Count on H_{p^3}-extensions is critical for other groups too:

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- $\nu(A_\ell \bullet \mathbb{Z}/p) = p^2 - 1$ for $2 < \ell < p$
- $\nu\left(A_p^{\oplus k} \rtimes G\right) \geq p^k$

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Several automatic realizations for small 2-groups

- $Q_8 \Rightarrow Q_8 \curlywedge D_4 \Rightarrow D_4$
- $Q_{16} \Rightarrow \mathbb{Z}/4$
- $SD_{16} \Rightarrow \mathbb{Z}/4$
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- $H_{p^3} \Rightarrow M_{p^3}$

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Fewer for $p > 2$:

- $\mathbb{Z}/p^{a_1} \times \mathbb{Z}/p^{a_2} \Rightarrow \mathbb{Z}/p^{b_1} \times \mathbb{Z}/p^{b_2}$ iff $\min(a_1, a_2) \geq \min(b_1, b_2)$
- $H_{p^3} \Rightarrow M_{p^3} \Rightarrow M_{p^3} \curlywedge \mathbb{Z}/p^2$
- $H_{p^3} \times K \Rightarrow M_{p^3} \times K$ for any finite group K

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Example: a group of size 3125 $\Rightarrow$ a group of size 48828125