Non-totally real cubic number fields and Cousin groups

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Goal

Study the connection between certain commutative connected complex Lie groups of the form $\mathbb{C}^2/\Gamma$, where Γ is a lattice of rank 3 in $\mathbb{C}^2$, and non-totally real cubic number fields.

Outline

- Motivation.
- Commutative connected complex Lie groups.
- The Remmert-Morimoto decomposition.
- Non-totally real cubic number fields and Cousin groups of complex dimension 2 and rank 3.
- Linearization of systems of exponents.

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Motivation

Let K be a CM-field of degree $2n$ over $\mathbb{Q}$ and let

$$\Phi = \{\varphi_1, \ldots, \varphi_n\}$$

be a CM-type. Define $\mu_\Phi : K \longrightarrow \mathbb{C}^n$ by the formula

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Motivation

If $\mathfrak{a}$ is a fractional ideal of K, then $\mu_\Phi(\mathfrak{a})$ is a lattice of full rank in $\mathbb{C}^n$ and the complex torus

$$\mathbb{C}^n/\mu_\Phi(\mathfrak{a})$$

admits a Riemann form. Therefore, these complex tori can be embedded as abelian varieties inside some projective space $\mathbb{P}^m(\mathbb{C})$. Moreover, their rings of endomorphisms are "big". For example, if the corresponding abelian variety A is simple, then

$$\mathrm{End}_{\mathbb{C}}(A) \simeq O_K.$$

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The case $n = 1$ might be more familiar: The complex tori are the elliptic curves with complex multiplication and one has the following theorem:

Theorem

There are exactly h_K isomorphism classes of elliptic curves with CM by O_K.

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Motivation

Question: How essential is the compactness of the complex tori???

Motivation

For instance, one has

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{G}_m(\mathbb{C})$$

via $z \mapsto \exp(2\pi iz)$.

Commutative connected complex Lie groups

Theorem

If G is a commutative connected complex Lie group of complex dimension n, then there exists a lattice Γ (not necessarily of full rank) such that

$$G \simeq \mathbb{C}^n/\Gamma.$$

The Remmert-Morimoto decomposition

Theorem (Remmert ?, Morimoto 1965)

Any commutative connected complex Lie group is isomorphic to a group of the form

$$\mathbb{C}^a \times (\mathbb{C}^\times)^b \times G_0,$$

where G_0 is a commutative connected complex Lie group satisfying $\mathrm{Hol}(G_0) = \mathbb{C}$. Moreover, this decomposition is unique meaning that if

$$\mathbb{C}^a \times (\mathbb{C}^\times)^b \times G_0 \simeq \mathbb{C}^{a'} \times (\mathbb{C}^\times)^{b'} \times G_0',$$

then

$$a = a', \ b = b', \text{ and } G_0 \simeq G_0'.$$

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Definition

A commutative connected complex Lie group G is called a Cousin group if $\mathrm{Hol}(G) = \mathbb{C}$.

Non-totally real cubic number fields and Cousin groups

Let K be a number field, $r = r_1 + r_2$, and

$$\Phi = \{\varphi_1, \ldots, \varphi_r\}$$

be a complete set of representatives modulo complex conjugation for the embeddings of K into $\mathbb{C}$, where the first r_1 embeddings are real. We define $\mu_\Phi : K \longrightarrow \mathbb{C}^r$ by the formula

$$\lambda \mapsto (\varphi_1(\lambda), \ldots, \varphi_r(\lambda)).$$

Then given any fractional ideal $\mathfrak{a}$ of K, $\mu_\Phi(\mathfrak{a})$ is a lattice of rank $[K : \mathbb{Q}]$ and one can look at the commutative connected complex Lie group

$$\mathbb{C}^r / \mu_\Phi(\mathfrak{a}).$$

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If K is totally real then

$$\mathbb{C}^r/\mu_\Phi(\mathfrak{a}) \simeq (\mathbb{C}^\times)^{r_1}.$$

Theorem (Gherardelli 1989, V.)

If K is a non-totally real cubic number field, then

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is a Cousin group.

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F. Gherardelli, Varieta' quasi abeliane a moltiplicazione complessa, Rendiconti del Seminario Matematico e Fisico di Milano, 1989.
Student: Giorgio Ottaviani

Non-totally real cubic number fields and Cousin groups

If G is a Cousin group of complex dimension 2 and rank 3, then most of the time

$$\mathrm{End}(G) = \mathbb{Z}.$$

Theorem (Gherardelli 1989, V.)

Let G be a Cousin group of complex dimension 2 and of rank 3. If

$$\mathrm{End}(G) \neq \mathbb{Z},$$

then

$$\mathrm{End}_0(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(G)$$

is a non-totally real cubic number field and $\mathrm{End}(G)$ is an order.

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Let K be a non-totally real cubic number field and let $\Phi = \{\varphi_1, \varphi_2\}$ be as before. If $\mathfrak{a}$ is any fractional ideal of K, then

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is a Cousin group satisfying

$$\mathrm{End}\left(\mathbb{C}^2/\mu_\Phi(\mathfrak{a})\right) \simeq O_K.$$

In fact, any Cousin group of complex dimension 2 and rank 3 having "extra multiplication" by O_K is isomorphic to one of those.

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Definition

Suppose that G is a Cousin group of complex dimension 2 and rank 3 satisfying

$$\iota : K \xrightarrow{\simeq} \mathrm{End}_0(G).$$

Then, $\rho_a \circ \iota \simeq \varphi_1 \oplus \varphi_2$, where φ_1 is the unique real embedding and φ_2 is one of the two complex embeddings. We then say that (G, ι) is of type (K, Φ), where

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Let Σ_Φ be the set of isomorphism classes of Cousin groups (G, ι) of type (K, Φ).

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One defines an action $Cl_K \times \Sigma_\Phi \longrightarrow \Sigma_\Phi$ via

$$[\mathfrak{a}] \cdot \left[\mathbb{C}^2/\mu_\Phi(\mathfrak{b})\right] \mapsto \left[\mathbb{C}^2/\mu_\Phi(\mathfrak{a}\mathfrak{b})\right].$$

Theorem (Gherardelli 1989, V.)

This action is simply transitive and therefore there are exactly h_K isomorphism classes of Cousin groups (G, ι) of type (K, Φ).

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Linearization of systems of exponents

In general, one can study $\mathcal{M}(\mathbb{C}^n/\Gamma)$, the field of meromorphic functions on these complex manifolds. Such an $f \in \mathcal{M}(\mathbb{C}^n/\Gamma)$ can be written as

$$f = \frac{g_1}{g_2},$$

for some $g_1, g_2 \in \mathrm{Hol}(\mathbb{C}^n)$, where the g_i satisfy a certain functional equation involving a system of exponents.

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Recall what a system of exponents is: It is a map $s : \Gamma \times \mathbb{C}^n \longrightarrow \mathbb{C}$ satisfying

- The map $s_\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $z \mapsto s(\gamma, z)$ is holomorphic.
- $s(0, z) \in \mathbb{Z}$ for all $z \in \mathbb{C}^n$.
- $s(\gamma + \gamma', z) - (s(\gamma, z + \gamma') + s(\gamma', z)) \in \mathbb{Z}$.

The functional equation satisfied by the g_i is

$$g_i(z + \gamma) = \exp\left(2\pi i s(\gamma, z)\right) \cdot g_i(z).$$

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Two systems of exponents s and s' are equivalent if and only if there exists $d \in \mathrm{Hol}(\mathbb{C}^n)$ satisfying

$$\big(s'(\gamma, z) - s(\gamma, z)\big) + \mathbb{Z} = (d(z + \gamma) - d(z)) + \mathbb{Z}.$$

Linearization of systems of exponents

Question: Can one find g_i with a simplest possible system of exponents, i.e. of the form

$$s(\gamma, z) = L_\gamma(z) + c(\gamma),$$

where $L_\gamma \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ and $c(\gamma) \in \mathbb{C}$. If yes, s is said to be linearizable. A function g satisfying

$$g(z+\gamma) = \exp\left(2\pi i \left(L_\gamma(z) + c(\gamma)\right)\right) \cdot g(z)$$

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Remarks:

- In the compact case, it is always possible and this is the Appell-Humbert theorem.
- P. Cousin (Sur les fonctions triplement périodiques de deux variables, Acta. Math., 1910) gave counter-examples for certain $\mathbb{C}^2/\Gamma$ where Γ has rank 3.
- C. Vogt (Line bundles on toroidal groups, Crelle, 1982) characterizes Cousin groups for which systems of exponents are always linearizable.

Theorem (V.)

Let $\mathbb{C}^2/\Gamma$ be a Cousin group of complex dimension 2 and rank 3 having "extra multiplication". Then, any system of exponents is linearizable, i.e. every Γ-periodic function can be written as a quotient of theta functions.

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