Relative Bogomolov Extensions

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Bobby Grizzard

Department of Mathematics
The University of Texas at Austin
`rgrizzard@math.utexas.edu`

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The absolute height of algebraic numbers

If $\alpha \in \overline{\mathbb{Q}}$ has minimal polynomial

$$f(x) = a_0 x^d + \cdots + a_d = a_0(x - \alpha_1)\cdots(x - \alpha_d) \in \mathbb{Z}[x],$$

we define

$$H(\alpha) = \underbrace{\left(a_0 \prod_{i=1}^{d} \max\{1, |\alpha_i|\} \right)}_{M(f) = \text{“Mahler measure of } f\text{”}}^{\frac{1}{d}} = \prod_{v} \max\{1, |\alpha|_v\}.$$

- The last product is over all places of an arbitrary number field $K \ni \alpha$.
- H is the “multiplicative” height; $h := \log H$ denotes the “logarithmic” or “additive” height.

Properties of $H : \overline{\mathbb{Q}}^{\times} \to [1, \infty)$

- $H(\alpha) = 1$ iff α is a root of unity, a.k.a. a torsion point of $\mathbb{G}_m(\overline{\mathbb{Q}})$.
- If $\alpha = p/q \in \mathbb{Q}, (p, q) = 1$, then $H(\alpha) = \max\{|p|, |q|\}$.
- Galois-invariant (all roots of same irred. poly. have same height)
- If $\lambda \in \mathbb{Q}$, then $H(\alpha^{\lambda}) = H(\alpha)^{|\lambda|}$ ("scaling")
- $H(\alpha\beta) \leq H(\alpha)H(\beta)$ ("triangle inequality")
- Roughly comparable to ℓ^1- and ℓ^{∞}-norms of coefficients of minimal polynomial, which are easier to compute but don't play nice.

Other heights

- Absolute height on algebraic numbers $h : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty)$ (the focus for this talk)– more generally on $\mathbb{G}_m^n$ sum the heights of coordinates.
- Absolute height of a point in projective space $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0, \infty)$ is just a slight adjustment of the previous definition.
- If $V/\overline{\mathbb{Q}}$ is a variety and we have a map $f : V \to \mathbb{P}^n$, this induces a height $h_f : V(\overline{\mathbb{Q}}) \to [0, \infty)$, by $h_f(P) = h(f(P))$.
- For an abelian variety A we have the Néron-Tate canonical height $\hat{h} : A(\overline{\mathbb{Q}}) \to [0, \infty)$, which is "close" ($O(1)$ away) to h_f for any f and respects the group structure and Galois action, analogous to h on $\mathbb{G}_m$.
- Fancy results in diophantine geometry (e.g. Faltings's Theorem) use fancier heights.

Unconditional lower bounds – the Lehmer conjecture

Conjecture (D. H. Lehmer, '33)

There exists an absolute constant $c > 1$ such that if α is an algebraic number of degree d over $\mathbb{Q}$, not a root of unity, then

$$H(\alpha)^d \geq c.$$

- Evidence (ask Mike Mossinghoff) suggests that we can take $c = 1.17628...$, achieved when α is a root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, already discovered by Lehmer in '33 (calculating by hand!).
- Dobrowolski ('79) showed we can get $H(\alpha)^d \geq c' \cdot \left(\frac{\log\log d}{\log d}\right)^3$, and Voutier later showed here we can take $c' = 1/4$ for all $d > 1$.

Bounds for subfields of $\overline{\mathbb{Q}}$ – the Bogomolov property

Definition (Bombieri & Zannier, '01)

A subfield $K \subseteq \overline{\mathbb{Q}}$ satisfies the *Bogomolov property* if there exists $\varepsilon > 0$ such that there is no non-torsion point $\alpha \in K^\times$ with $h(\alpha) < \varepsilon$.
("K has no small points.")

- Easy to see all number fields have (B).
- We say K has (B) w.r.t. an abelian variety A if $A(K)$ has no small points in the canonical height.
- Named after the related Bogomolov conjectures in diophantine geometry.
- K has (B) $\Leftrightarrow$ $K^\times$/tors is a discrete subgroup of $\overline{\mathbb{Q}}^\times$/tors.
- This means that if K has (B), then $K^\times$/tors is a free abelian group. Same for $A(K)$/tors if K has (B) w.r.t. A.

Fields with (B) – no small points

- $\mathbb{Q}^{\text{tot. real}}$ (Schinzel '73) – α tot. real $\Rightarrow H(\alpha) \geq \frac{1+\sqrt{5}}{2}$ (sharp)
- $\mathbb{Q}^{ab}$ (Amoroso & Dvornicich '00)
- If we interpret the above as a result about heights on $\mathbb{G}_m(\mathbb{Q}(\mathbb{G}_m, \text{tors}))$, this generalizes in several ways:
    - $\mathbb{G}_m(k^{ab})$, k a number field (Amoroso & Zannier, '00, '10)
    - $A(\mathbb{Q}(\mathbb{G}_{m,\text{tors}}))$, $A/\mathbb{Q}$ an abelian variety (Baker & Silverman '04)
    - $\mathbb{G}_m(\mathbb{Q}(E_{\text{tors}}))$ and $E(\mathbb{Q}(E_{\text{tors}}))$, $E/\mathbb{Q}$ an elliptic curve (Habegger '13)
- Any Galois extension of $\mathbb{Q}$ which embeds into a finite extension of $\mathbb{Q}_p$ for some p (i.e. "totaly $\mathfrak{p}$-adic" (Bombieri & Zannier '01)
- Any extension L of a number field k such that $\text{Gal}(L/k)/Z(\text{Gal}(L/k))$ has finite exponent (Amroso, David, & Zannier, '13)

The relative Bogomolov property

Definition

Let $K \subseteq L$ be subfields of $\overline{\mathbb{Q}}$. The extension L/K is *Bogomolov* (or satisfies the *relative Bogomolov property*, (RB)) if there exists $\varepsilon > 0$ such that no non-torsion point $\alpha \in L^\times \setminus K^\times$ has $h(\alpha) < \epsilon$.
("L has no new small points.")

- If K has (B), then L/K has (RB) iff L has (B).
- For $M/L/K$, M/K is (RB) iff M/L and L/K are both (RB).
- If $L \setminus K$ has a root of unity and K has small points, so does $L \setminus K$.

Theorem (G., Pottmeyer (indep.))

This can happen even when K has small points.

Examples

1. Let $K = \mathbb{Q}(\sqrt[3]{3}, \sqrt[9]{3}, \sqrt[27]{3}, \dots)$ (choosing always the real root).
   - $h(3^{1/3^n}) = \frac{1}{3^n} \cdot h(3) = \frac{\log 3}{3^n} \to 0$ as $n \to \infty$, so K does not have (B).
   - Note that the only proper subfields of K are the finite extensions $\mathbb{Q}(\sqrt[n]{3})$, $n = 0, 1, 2, \dots$. Since all proper subfields are finite, K does not have (RB) over any subfield.
   - $K(\sqrt{3})$ does not have (RB) – intuitively, K has points that are "close" to $\sqrt{3}$.
   - $K(\sqrt{p})$ *does* have (RB) for p any other prime – this is harder to see, but a key fact is that p is unramified in K.
2. Let $K = \mathbb{Q}^{\text{tot.real}}(\sqrt{-1})$ (the "maximal CM field").
   - K has small points (Amoroso & Nuccio '07), even though $\mathbb{Q}^{\text{tot.real}}$ does not (Schinzel).
   - There is no extension L/K having (RB) (Pottmeyer, '13).

Main result

Theorem (G.)

Let $K/\mathbb{Q}$ be a Galois extension. If there is a (finite) rational prime p with bounded ramification index in K, then there exist relative Bogomolov extensions L/K.

- These extensions are of the form $K(\sqrt[p]{\alpha})$ for an appropriate choice of $\alpha \in K$.
- cf. Bombieri & Zannier: if K has bounded local degree (ram. index times residual degree) at some prime, then K has (B).

Proof ingredients:

- A "non-archimedean" inequality of Silverman ('84) which bounds from below the heights of generators of relative extensions in terms of the relative discriminant (this generalizes a classical bound of Mahler).
- An "archimedean" bound of Garza ('07) which generalizes Schinzel's theorem for totally real numbers.

It's over!