Introduction to Green functions and theta lifting identities

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What is a Green function?

In simple words a Green function G is the solution to an equation which looks like:

$$\boxed{D G \overset{(*)}{=} \delta}$$

where
1) δ is a Dirac distribution
2) D is a linear partial differential operator which usually involves a "Laplacian like operator".
3) G is the unique solution to $(*)$ which satisfies some "boundary conditions".

Let (M,g) be a Riemannian manifold

$x, y \in M$ be variables

Δ_x = Laplacian in the x variable $\curvearrowright L^2(M)$

$\hookrightarrow$ positive Laplacian so in $\mathbb{R}^n$ $\quad \Delta_x = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

Examples of Green functions

(1) Let $\quad D = \frac{\partial}{\partial t} + \Delta_x \quad$ and $\quad G = U^M(x,y;t)$ be the

unique solution to

$$\boxed{\begin{gathered}\left(\frac{\partial}{\partial t} + \Delta_x\right) U^M(x,y;t) = \delta_y(x)\cdot\delta(t) \\ t>0, \quad x,y \in M\end{gathered}}$$

$U^M(x,y;t)$ is called the <u>heat kernel of M</u>

(a) If M is compact then the spectrum of $\Delta_x \circlearrowleft L^2(M)$ is discrete; say : $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_k \leq \cdots$

Then $$U^M(x,y;t) = \sum_{k \geq 1} u_k(x)\,\overline{u_k(y)}\, e^{-\lambda_k t}$$ where

$\{u_k(x)\}_{k \geq 1}$ is an orthonormal basis of eigenvectors of Δ_x.

(b) if $M = \mathbb{R}^n$ then spectrum $(\Delta_x) = [0,\infty)$ is absolutely continuous

and $$U^{\mathbb{R}^n}(x,y;t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}$$

where $r = |x-y|$

(c) In general, if (M_1, g_1) and (M_2, g_2) are Riemannian manifolds we always have

$$\boxed{\begin{gathered} U^{M_1 \times M_2}\Big((x_1, x_2), (y_1, y_2) ; t \Big) = U^{M_1}(x_1, y_1; t) \cdot U^{M_2}(x_2, y_2; t) \\ (x_1, x_2), (y_1, y_2) \in M_1 \times M_2 \end{gathered}}$$

2. $D = \Delta_x - \lambda$ where λ is a spectral parameter

Let $G = G^M(x,y;\lambda)$ be the unique solution to

$$\boxed{(\Delta_x - \lambda)\, G^M(x,y;\lambda) \overset{(\dagger)}{=} \delta_y(x).}$$

$\hookrightarrow$ resolvent Green function of $\Delta_x \hookrightarrow L^2(M)$

It follows from $(\dagger)$ that $G^M(x,y;\lambda)$ satisfies the following reproducing identity

$$\boxed{(\Delta_x - \lambda) \int_M G^M(x,y;\lambda)\, f(y)\, dy = f(x) \quad \text{for } f \in L^2(M) \text{ and } \lambda \notin \sigma(\Delta_x)}$$

⑥

a) If M is <u>compact</u> then

$$\boxed{G^{M}(x,y;\lambda) = \sum_{k\geq 1} \frac{u_k(x)\overline{u_k(y)}}{\lambda-\lambda_k} \quad \text{where } \lambda \in \mathbb{C},\ \operatorname{Re}(\lambda)<0}$$

where $\{u_k(x)\}_{k\geq 1}$ is an othonormal basis of eigenvectors of $\Delta_x \circlearrowleft L^2(M)$.

b) If $M = \mathbb{R}^n$ then

$$\boxed{G^{\mathbb{R}^n}(x,y;\lambda) = (2\pi)^{-n/2}\, \lambda^{\frac{n}{4}-\frac{1}{2}}\, r^{1-\frac{n}{2}}\, K_{\frac{n}{2}-1}(r\sqrt{\lambda}) \quad \text{where } r = |x-y|}$$

↳ K-Bessel function

In particular, $G^{\mathbb{R}^n}(x,y;\lambda) \underset{\text{as } r\to 0}{\sim} \begin{cases} \dfrac{-1}{2\pi}\log r & \text{if } n=2 \\[2ex] \dfrac{1}{n(n-2)\operatorname{vol}(B_n)} \cdot \dfrac{1}{r^{n-2}} & \text{if } n\geq 3 \end{cases}$

The resolvent Green function $G^M(x,y;\lambda)$ and the heat kernel are related in the following way: (7)

$$G^M(x,y;\lambda) = \int_0^\infty U^M(x,y;t)\, e^{\lambda t}\, dt \qquad \mathrm{Re}(\lambda) \ll 0$$

and

$$U^M(x,y;t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G^M(x,y;\lambda)\, e^{-\lambda t}\, d\lambda \qquad \text{for } t>0, \quad \mathrm{Re}(c) \ll 0$$

c) Consider the space $(M, g) = \left(h,\ ds^2 = \frac{dx^2+dy^2}{y^2}\right)$ where (8)

$h = \{ x+iy \in \mathbb{C} : y > 0 \}$ Poincaré upper half-plane

Let $G_s^h(z_1,z_2) := G^h(z_1,z_2;\ s(1-s))$

$\hookrightarrow$ resolvent Green function on h

$$\Delta_{z_1} = -y_1^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \quad \text{where} \quad \begin{aligned} z_1 &= x_1 + iy_1 \\ z_2 &= x_2 + iy_2 \end{aligned}$$

$\circlearrowright L^2(h)$ spectrum$(\Delta_{z_1}) = [\frac{1}{4}, \infty)$ and is absolutely continuous

We have
$$\boxed{G_s^h(z_1,z_2) = \frac{\Gamma(s)^2}{4\pi\,\Gamma(2s)}\left(1 - \mathrm{th}^2\tfrac{\rho}{2}\right)^s {}_2F_1\left(s, s, 2s;\ 1 - \mathrm{th}^2\tfrac{\rho}{2}\right) \\ \text{where} \quad \rho = \mathrm{dist}_{hyp}(z_1,z_2) \qquad \mathrm{th}\tfrac{\rho}{2} = \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|}$$

Let $\Gamma \leq SL_2(\mathbb{R})$ be a discrete subgroup and let

$Y := \Gamma \backslash h$ (hyperbolic surface)

Then

$$\boxed{G_s^Y(z_1, z_2) := \sum_{\gamma \in \Gamma} G_s^h(z_1, \gamma z_2) \qquad Re(s) > 1}$$

$\hookrightarrow$ just the Γ-group average of the resolvent Green function on h.

An example: Let $\Gamma = SL_2(\mathbb{Z})$ $Y = SL_2(\mathbb{Z}) \backslash \mathfrak{h}$

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \qquad \mathrm{Re}(s) > 1.$$

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = y^s + \frac{\zeta(2s-1)}{\zeta(2s)} \varkappa(s) y^{1-s} + O\left(\exp(-cy)\right)$$

$z = x + iy \in \mathfrak{h}$, $\mathrm{Re}(s) > 1$.

$$G_s^Y(z_1, z_2) = \frac{-y_2^{1-s}}{1-2s} E(z_1, s) + O\left(\exp(-c y_2)\right).$$

$\longrightarrow$ Fourier series expansion of $\left[x_2 \longmapsto G_s^Y(z_1, x_2 + i y_2)\right]$

"Resolvent Green function's Kronecker limit formula"

$$Y = SL_2(\mathbb{Z}) \backslash h$$

$$\lim_{s \to 1} G_s^Y(z_1, z_2) \overset{(*)}{=} \frac{-\frac{3}{\pi}}{s(1-s)} - \frac{1}{2\pi} \log \left| y_1^6 y_2^6 P(z_1, z_2) \right| + O(s-1)$$

where $P(z_1, z_2) = \left(j(z_1) - j(z_2) \right) \Delta(z_1) \Delta(z_2)$

↳ "Selberg prime form of Y"

It is not too difficult to deduce from (*) the First Kronecker limit Formula for $E(z,s)$ as $s \to 1$

The Jacquet Langlands correspondance

Let $B_1 = M_2(\mathbb{Z})$

Let $B_2 = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ where $i^2 = q$, $j^2 = r$ where r, q are positive square-free and coprime integers such that
$1 = qx^2 + ry^2$ has no rational solution.

It follows that B_2 is an <u>order</u> of an <u>indefinite division algebra</u> where the discriminant of the order is $(4qr)^2$.

Let $\Gamma_1 := \Gamma_0(4qr) \leqslant B_1^\times$ and $\Gamma_2 =$ reduced norm 1 elements of B_2

Let $Y_j := \Gamma_j \backslash \mathfrak{h}$ for $j = 1, 2$.

Y_1 is <u>non-compact</u> while Y_2 is <u>compact</u>.

The surfaces Y_1 and Y_2 are "arithmetically close" from one another since $B_1 \otimes_{\mathbb{Z}} \mathbb{Q}(\sqrt{q}, \sqrt{r}) \simeq B_2 \otimes_{\mathbb{Z}} \mathbb{Q}(\sqrt{q}, \sqrt{r})$.

<u>Key fact</u>: It follows from Jacquet-Langlands correspondance (or the Selberg trace formula) that

$$\left\{ \text{spectrum } \Delta \circlearrowleft L^2(Y_2) \right\} \subseteq \left\{ \begin{array}{l} \text{discrete} \\ \text{spectrum} \end{array} \Delta \circlearrowleft L^2(Y_1) \right\}$$

Theta lifting identity

Let $\mathcal{F}_j \subseteq h$ be a fundamental domain for Γ_j $(j = 1, 2)$

It was noticed by John Fay that such a spectral correspondance "is manifested" through the following "theta lifting identity":

$$\int_{\mathcal{F}_1} \Theta(\tau, z_0)\, G_s^{\gamma_1}(\tau, z)\, d\mu(\tau) = \int_{\mathcal{F}_2} \Theta(z, w)\, G_s^{\gamma_2}(z_0, w)\, d\mu(w)$$

where $\Theta(\tau, z)$ is a suitable Siegel-theta function

and $d\mu(\tau) = \frac{du\, dv}{v^2}$ if $\tau = u + iv$

The proof uses 2 ingredients:

1. $\Delta_z \Theta(z,w) = \Delta_w \Theta(z,w)$

2. reproducing property of $G_s^{Y_j}(z,w)$ for $j=1,2$