The Zariski closure of integral points arising from periodic continued fractions

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Reminders on continued fractions

A *continued fraction* is an expression of the form $a_1 + 1/(a_2 + 1/(\dots))$ for a_i in a fixed integral domain R.

If the continued fraction is finite, it defines an element of the fraction field of R.

Every element of $\mathbb{Q}^+$ has a finite continued fraction with $a_1 \in \mathbb{N}$ and $a_i \in \mathbb{Z}^+$ for $i > 1$, which is unique if we assume that the last a_i is not 1.

Every $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ has a unique continued fraction with $a_1 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}^+$ for $i > 1$ which is *eventually periodic*: i.e., $a_{n+c} = a_n$ for sufficiently large n.

Square roots

We write $[a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_n}]$ for the continued fraction for the sequence $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n, a_{k+1}, \ldots$.

For R a general ring, there is no analogue of $\mathbb{N} \subset \mathbb{Z}$ and no canonical concept of convergence in the fraction field.

A periodic continued fraction over R defines a quadratic with coefficients in R by writing

$$x = [\overline{a_{k+1}, \ldots, a_n}] = [a_{k+1}, \ldots, a_n, \overline{a_{k+1}, \ldots, a_n}]$$

and rearranging to get $x = a + b/x$.

In this talk we are interested in continued fractions for $\sqrt{t}$: in other words, continued fractions whose quadratic is $x^2 - t$.

Example

Consider the continued fraction $[1, \overline{2}]$. Let this be α: we then have $1 + 1/(\alpha + 1) = \alpha$. Multiplying through by $\alpha + 1$ we obtain $\alpha + 2 = \alpha^2 + \alpha$ or $\alpha^2 - 2 = 0$. This is the usual continued fraction for $\sqrt{2}$. However, if we permit the a_i to be negative, other eventually periodic continued fractions exist. For example, $[3, \overline{-1, 2, 3}]$ also defines the polynomial $x^2 - 2$.

Can we classify all eventually periodic continued fractions of a given length (with coefficients in $\mathbb{Z}$ or some other ring) that define this polynomial?

Varieties

Fix $k, n \in \mathbb{Z}$ and $t \in R$ and consider continued fractions $C : [a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_n}]$.

The coefficients of the quadratic polynomial defined by C are polynomials in the a_i.

Thus the condition for C to correspond to $x^2 - t$ defines a subvariety of codimension 2 in $\mathbb{A}^n(R)$. We write $S_{k,n-k,t}$ for this variety.

We ask the question: are the R-points of $S_{k,n-k,t}$ Zariski dense? If not, do they become so over a finite integral extension of R, or after inverting finitely many elements of R?

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Curves

The following result is quite famous:

Theorem

(Faltings) Let C be a curve over a number field K. Then the rational points of C are Zariski-dense over some finite extension of K (we say "potentially dense") if and only if $g_C \leq 1$.

A result on integral points is also well-known:

Theorem

(Siegel) Let C be an affine curve over a number field K. Then the integral points on C are potentially dense if and only if $g_C = 0$ and C has at most 2 points at infinity.

Higher dimensions

The question of potential density of rational points is complicated. The canonical divisor is expected to have a strong influence.

Conjecture

1. *(Lang) Let V be a variety such that, for some $n > 0$, the map defined by the linear system $|nK_V|$ is a birational equivalence. Then rational points on V are not Zariski dense.*
2. *Let V be a variety such that $|-nK_V|$ defines a birational equivalence for some $n > 0$. Then rational points on V are potentially dense.*
3. *Let V be a Calabi-Yau variety. Then rational points on V are potentially dense.*

Intermediate situations are not well-understood, though ideas of Campana offer hope.

Logarithmic geometry

On a proper variety, integral and rational points are the same.

In general there is the following idea:

Principle

(Iitaka) Let $P(V)$ be a property of varieties that is governed by the canonical divisor for V proper. Then $P(V)$ is governed by the log canonical divisor in general.

(Given non-proper V, let $\bar{V}$ be a proper variety such that $D = \bar{V} \setminus V$ is a union of reduced divisors such that all singularities look locally like $x_1 x_2 \dots x_i = 0$. Then the log canonical divisor of V is defined to be $K_{\bar{V}} + D$. $\bar{V}$ is not unique; part of the principle is that the choice does not matter.)

Examples

Potential density is such a property. For curves, the conjectures are theorems of Siegel and Faltings.

In higher dimensions, much less is known, but there are results in both directions. For example, integral points are not dense on semiabelian varieties of log general type. Also, etale covers preserve both potential density and these conjectures.

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Plan for the next few minutes

We now return to the varieties $S_{k,n-k,t}$ that parametrize continued fractions for $\sqrt{t}$ with preperiodic part of length k and period of length $n-k$. We will be most interested in the cases $n=4$, in which $\dim S = 2$.

We start by discussing $S_{0,4,t}$. It turns out that $S_{1,3,t}$ is strikingly similar to $S_{0,4,t}$, and more generally that $S_{1,n-1,t}$ is very much like $S_{0,n,t}$. We do not understand this.

We will determine a suitable compactification and study the log canonical divisor.

Compactifying $S_{0,4,t}$

Since $S_{0,4,t} \subset \mathbb{A}^4$, the first thing to do is to look at its Zariski closure $P_{0,4,t} \subset \mathbb{P}^4$. Let $p_1, \ldots, p_5$ be the coordinates on $\mathbb{P}^4$.

The divisor at infinity (i.e., $p_5 = 0$) consists of 4 double lines, along each of which $P_{0,4,t}$ is singular.

Let $C_{0,4,t}$ be the normalization in $\mathbb{P}^6$. It is a surface with 5 ordinary double points. Letting $q_1, \ldots, q_7$ be the coordinates on $\mathbb{P}^6$, the divisor at infinity is defined by $q_7 = 0$ (because q_7 is the only coordinate that involves p_5).

The snc compactification

The divisor $q_7 = 0$ on $C_{0,4,t}$ has 6 components, any 2 meeting transversely and no 3 intersecting.

Also, 4 of the singular points have $q_7 = 0$. Their strict transforms should also be considered as boundary divisors.

Let $\tilde{C}_{0,4,t}$ be the minimal desingularization of $C_{0,4,t}$. Then the divisor $q_7 = 0$ on $\tilde{C}_{0,4,t}$ is a simple normal crossings divisor (i.e., the singularities are locally of the form $x_1 x_2 = 0$).

The simple model

Definition

Let S be a smooth surface and D an snc divisor on S. Suppose that S has no smooth rational curve E with $E^2 = -1$ and E contained in or disjoint from D. Then (S, D) is *simple*.

We may always blow down curves on (S, D) to obtain a simple model. This does not affect potential density of integral points.

(Blowing down a general -1-curve E might. A point that misses the boundary but meets E mod p meets the boundary mod p after E is contracted.)

The simple model of $\tilde{C}_{0,4,t}$ is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point and an infinitely near point.

The log Iitaka fibration

For a proper variety V, the *Iitaka fibration* is the map $V \to \mathbb{P}^k$ given by $|nK_V|$ for sufficiently large and divisible n. (The dimension of its image is the Kodaira dimension.)

More generally, for a pair $(\bar{V}, D)$, the *log Iitaka fibration* is the map $\bar{V} \to \mathbb{P}^k$ given by $|n(K_{\bar{V}} + D)|$ for sufficiently large and divisible n. It is the analogue of the Iitaka fibration for $\bar{V}$.

Proposition

The log Iitaka fibration of the simple model of $\tilde{C}_{0,4,t}$ is a map to a conic whose general fibre is a conic.

(This means that the base and general fibre both have two points at infinity.)

A density result

We can use the log Iitaka fibration to prove that integral points on $\tilde{C}_{0,4,t}$ are potentially dense but not dense. Note that $\tilde{C}_{0,4,t}$ is analogous to an elliptic surface over an elliptic curve. Some but not all such surfaces have potentially dense rational points.

Proposition

The integral points on $\tilde{C}_{0,4,t}$ are potentially dense.

Proof.

(very brief sketch, after Levin-Yasufuku) A fibre in either direction is a conic. Extend to a nonreal field where integral points on one fibre are Zariski dense; then consider the fibres through these points in the other direction. □

Longer sketch of proof

Proof.
(sketch, following a paper of Levin-Yasufuku) There is a fibre of π_1 on $\tilde{C}_{0,4,t}$ that is a smooth rational curve meeting the boundary in 2 points and whose integral points are therefore potentially dense. Extend the ring so that they are dense and consider an integral point P on it: the fibre of π_2 through P is a smooth rational curve meeting the boundary in 2 points. It has an integral point, so it has infinitely many integral points over a field that is not totally real. Thus after a finite extension we have infinitely many curves on $\tilde{C}_{0,4,t}$ with infinitely many integral points. □

A non-density result

Proposition

The integral points on $\tilde{C}_{0,4,t}$ are not dense over $\mathbb{Z}$.

Proof.

(Proof 1) A classical result on continued fractions is that, for fixed k, n, there is at most one expression of $\alpha \in \mathbb{R}$ by a continued fraction with all $a_i \notin \{-2, -1, 0, 1, 2\}$. The result follows immediately (and indeed for all the $S_{k,n-k,t}$). □

This proof can be generalized to the ring of integers in an imaginary quadratic field. However, I find it slightly unsatisfactory.

The result can also be proved directly by elementary number theory, in a way that makes it easy to determine the integral points.

A second proof (short version)

Proof.

(Proof 2, sketch) The fibres of the log Iitaka fibration are conics which can be written down explicitly.
Only fibres above units of norm 1 in $K(\sqrt{t})$ can have integral points. As $a\sqrt{t} + b$ runs over such units, b/a approaches $\sqrt{t}$.
There are two types of points, one of which leads to b/a being close to an integer and close to $\sqrt{t}$, and one that leads to a/m approaching an irrational and $(a+1)/m$ being an integer, where m is the third coordinate.
Both of these can only hold finitely often. □

The first half of this works over number fields; the second half doesn't.

A second proof (long version)

Proof.

(Proof 2, sketch) We compose the map $S_{0,4,t} \to \tilde{C}_{0,4,t}$ with the log Iitaka fibration, obtaining a map ϕ_t whose fibre at $(x:1)$ is a conic. In fact x must be a unit of norm 1 in $K(\sqrt{t})$ (the boundary of the base of the fibration is $\{0,\infty\}$). If $x = a\sqrt{t}+b$, then the fibre at x is defined by

$$ap_3p_4 - bp_3p_5 + (a+1)p_5^2 = ap_1 - p_3 - btp_5 = p_2 - ap_4 + bp_5 = 0.$$

Put $p_5 = 1$ and assume that the $p_i \in \mathbb{Z}$: then $p_3|(a+1)$ from the first equation. If $p_3 = a+1$ then $a|(bt+1)$ from the second equation; but $b/a \to \sqrt{t}$, so this happens on finitely many fibres. Similarly if $p_3 = (a+1)/d$ for $d = -1, 2, -2$.
Otherwise let $\sqrt{t} = s + f$ with $s \in \mathbb{Z}$ and $|f| < 1/2$. Then one shows that $p_3 \to af$ and $a/p_3 \to 1/f$; but $(a+1)/p_3 \in \mathbb{Z}$ and f is irrational, so p_3 is bounded for fixed t. □

Remarks

1. For fixed $t \in \mathbb{N} \setminus \mathbb{N}^2$, it is easy to make the estimates in the proof explicit and list all purely periodic continued fractions of length 4 for $\sqrt{t}$.
2. The part of the argument that shows that integral points lie over units of relative norm 1 is valid over an arbitrary number field. However, when the units are not discrete in the real topology, it is not clear how to use this to draw conclusions about potential density: in particular, we do not know whether the integral points of $S_{0,4,t}$ are dense over totally real fields.

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Summary of this section

The integral points on $S_{2,2,t}$ and $S_{3,1,t}$ are not potentially dense (proof: they map to $\mathbb{P}^1 - 3$ points).

The Iitaka fibration on $S_{0,4,t}$ can be interpreted in terms of the Pell equation, and that seems to generalize to $S_{0,n,t}$ and $S_{1,n-1,t}$.

For $n = 5$ this can be used to prove non-density.

We conjecture that the same works for all $n > 4$ and that this fibration can also be used to prove potential density.

Other surfaces

As already noted, the behaviour of integral points on $S_{1,3,t}$ is very much like that on $S_{0,4,t}$, and the proofs are quite similar. For $S_{2,2,t}$ and $S_{3,1,t}$, the situation is quite different:

Proposition

The integral points on $S_{2,2,t}$ and $S_{3,1,t}$ are not potentially dense.

Proof.

Consider the maps to $\mathbb{P}^1$ given by $(a_1 : 1)$ and $(a_4 : 1)$ respectively. In both cases the image misses 3 points and so has only finitely many integral points over any finitely generated subring of a number field. Thus the integral points of the surfaces are contained in a finite union of fibres. □

The "Pell equation"

The convergent of the continued fraction for $\sqrt{t}$ just before the beginning of the periodic part gives the fundamental solution of the so-called Pell equation $y^2 - tx^2 = 1$.

It turns out that the Iitaka fibration on $S_{0,4,t}$ essentially coincides with the map $(N : D : 1)$ where
$N = a_1a_2a_3 + a_1 + a_3, D = a_2a_3 + 1$ are the numerator and denominator of this convergent. Similarly for $S_{1,3,t}$.

In general it is not easy to construct an snc compactification of $S_{k,n-k,t}$. However, for $k = 0, 1$, the image of the analogous map to $\mathbb{P}^2$ defined by $(N : D : 1)$ is a conic.

The log Iitaka fibration in dimension 3 . . .

For $k = 0, 1$ and $n = 5$, the fibres of these maps to conics are del Pezzo surfaces of degree 5.

Note that, as in the surface case, the fibre is a "log Calabi-Yau variety", that is, the class of a hyperplane section is $-K$.

These maps can be used to prove that integral points are not Zariski dense by writing the threefolds as conic bundles over a suitable surface. We expect that the argument that proves potential density on the surfaces could be adapted to this situation, and it is possible that the argument could be extended to $n > 5$.

We conjecture:

Conjecture

These maps to conics are the log Iitaka fibrations for $S_{0,5,t}$ and $S_{1,4,t}$.

. . . and higher dimensions?

Somewhat similarly, for $S_{0,6,t}$ the general fibre is a singular intersection of three quadrics in $\mathbb{P}^6$. It appears to have canonical singularities, so again we seem to have $K = -H$ and the fibre is a log Calabi-Yau (and has some interesting fibrations of its own).

Beyond dimension 4 the singularities at the boundary nest and interact in complicated ways and it would be very difficult to construct an snc compactification explicitly. Nevertheless we conjecture:

Conjecture

The maps to conics as above are the log Iitaka fibrations for $S_{0,n,t}$ *and* $S_{1,n-1,t}$ *for all* $n > 1$. *They can be used to prove that integral points are potentially dense but not dense on these varieties.*

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Thank you

Thank you for your attention.