The Arithmetic of Modular Grids

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Joint Work with M. Griffin, P. Jenkins

What is Zagier duality?

Let $f_{k,m}(z)$ be the unique weakly holomorphic modular form of weight k over $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1})$$

$$\begin{aligned}
f_{0,0}(z) &= 1 \\
f_{0,1}(z) &= q^{-1} + 196884q + 21493760q^2 + \dots \\
f_{0,2}(z) &= q^{-2} + 42987520q + 40491909396q^2 + \dots \\
\\
f_{2,1}(z) &= q^{-1} - 196884q - 42987520q^2 + \dots \\
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What is a modular form?

Let G and H be subgroups of $\mathrm{SL}_2(\mathbb{R})$

We say G and H are commensurable if

$$[G : G \cap H] < \infty \text{ and } [H : G \cap H] < \infty$$

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Define $\mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \}$

A function $f : \mathbb{H} \to \mathbb{C}$ is modular for Γ (of weight k with multiplier ν) if it is symmetric with respect to Γ

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If $f(z)$ is a weakly holomorphic modular form of weight k with multiplier ν, then we may write

$$f(z) = \sum_{n \gg -\infty} a_n q^n$$

where $q = e^{2\pi \mathrm{i} z}$

What is a modular form?

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and z arbitrary, define $\gamma z = \frac{az+b}{cz+d}$

Define $j(\gamma, z) = cz + d$

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A map $\nu : \Gamma \to \mathbb{C}^\times$ is a weight k multiplier if

$$\nu(\gamma_1)\nu(\gamma_2)j(\gamma_1, \gamma_2 z)^k j(\gamma_2, z)^k = \nu(\gamma_1\gamma_2)j(\gamma_1\gamma_2, z)^k$$

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A function $f : \mathbb{H} \to \mathbb{C}$ is modular of weight k for Γ with multiplier ν if

$$f(\gamma z) = \nu(\gamma)j(\gamma, z)^k f(z)$$

What is a modular form?

Definition

A weight k weakly holomorphic modular form is a function $f : \mathbb{H} \to \mathbb{C}$ such that:

- f is modular of weight k
- f is holomorphic
- f is meromorphic at its cusps $\Omega(\Gamma)$

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$M_k^!(\Gamma, \nu)$ = the space of weakly holomorphic forms

What is Zagier duality? (revisited)

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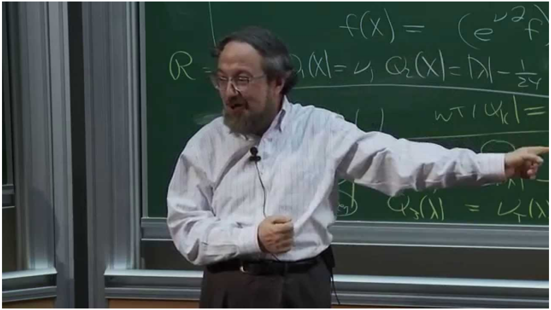
exhibit Zagier duality if

$$a(m,n) = -b(n,m)$$

Historical background



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Don Zagier published **Traces of Singular Moduli**

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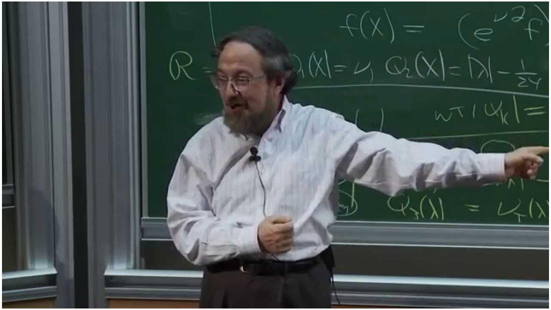

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He constructed bases for level 4 weakly holomorphic forms of weights 1/2 and 3/2 which satisfied the Kohnen plus space condition

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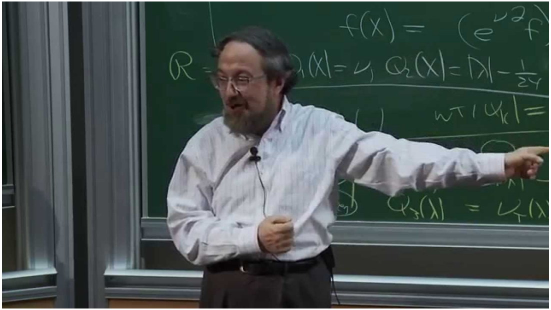

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$$a_{1/2}(m, n) = -a_{3/2}(n, m)$$

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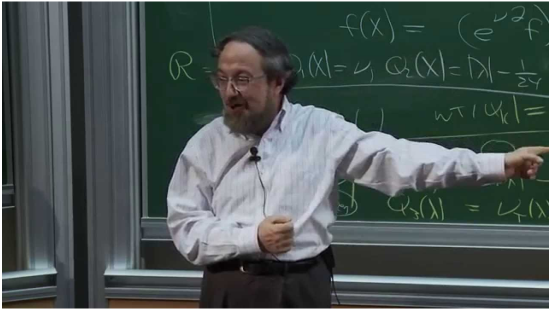

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One proof using recurrences

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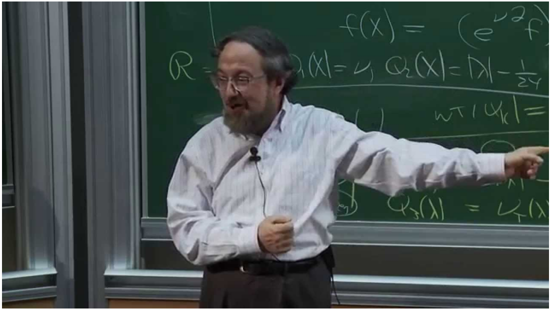

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One proof by observing the constant term of $f_m g_n$ is

$$a_{1/2}(m, n) + a_{3/2}(n, m) = 0$$

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1997

Tetsuya Asai, Masanobu Kaneko, and Hirohito Ninomiya published **Zeros of certain modular functions and an application**

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They proved duality between bases for level 1 spaces with weights k and $2 - k$ for $k \in \{ 0, 4, 6, 8, 10, 14 \}$

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Jeremy Rouse published **Zagier duality for the exponents of Borcherds products for Hilbert modular forms**

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2006

Jeremy Rouse published **Zagier duality for the exponents of Borcherds products for Hilbert modular forms**

He proved duality between bases for certain weight 0 and weight 2 spaces with nontrivial multipliers

Historical background

Bill Duke

Paul Jenkins

2007

Bill Duke and Paul Jenkins published **On the zeros and coefficients of certain weakly holomorphic modular forms**

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They constructed bases for level 1 weakly holomorphic modular forms of weights k and $2 - k$

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For all even k,

$$a_k(m, n) = -a_{2-k}(n, m)$$

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2008

Ahmad El-Guindy published **Fourier expansions with modular form coefficients**

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2013

SoYoung Choi and Chang Heon Kim published **Basis for the space of weakly holomorphic modular forms in higher level cases**

They extended Duke's and Jenkins' proof to establish duality between bases for forms over $\Gamma_0^+(p)$ with genus 0

Historical background

2014

Andrew Haddock and Paul Jenkins published **Zeros of weakly holomorphic modular forms of level 4**

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They extended Duke's and Jenkins' proof to establish duality between bases for level 4 forms of every even weight

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Victoria Iba, Paul Jenkins, and Merrill Warnick published **Congruences for coefficients of modular functions in genus zero levels**

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2017

Victoria Iba, Paul Jenkins, and Merrill Warnick published **Congruences for coefficients of modular functions in genus zero levels**

They extended Duke's and Jenkins' proof to establish duality between bases for forms with levels 6, 10, 12, 18 of every even weight

Historical background

2017

Daniel Adams published **Spaces of weakly holomorphic modular forms in level 52**

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Daniel Adams published **Spaces of weakly holomorphic modular forms in level 52**

He extended Duke's and Jenkins' proof to establish duality between bases for level 52 forms of every even weight

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Kit Vander Wilt published **Weakly holomorphic modular forms in level 64**

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Daniel Adams published **Spaces of weakly holomorphic modular forms in level 52**

He extended Duke's and Jenkins' proof to establish duality between bases for level 52 forms of every even weight

2017

Kit Vander Wilt published **Weakly holomorphic modular forms in level 64**

He extended Duke's and Jenkins' proof to establish duality between bases for level 64 forms of every even weight

Historical background

2017

Paul Jenkins and DJ Thornton published **Weakly holomorphic modular forms in prime power levels of genus zero**

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Paul Jenkins and DJ Thornton published **Weakly holomorphic modular forms in prime power levels of genus zero**

They extended Duke's and Jenkins' proof to establish duality between bases for forms with levels 2, 3, 4, 5, 7, 8, 9, 16, and 25, of every even weight

Historical background

2017

Paul Jenkins and DJ Thornton published **Weakly holomorphic modular forms in prime power levels of genus zero**

They extended Duke's and Jenkins' proof to establish duality between bases for forms with levels 2, 3, 4, 5, 7, 8, 9, 16, and 25, of every even weight

2017

Paul Jenkins and the author published **Zagier duality for level p weakly holomorphic modular forms**

Historical background

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They extended Duke's and Jenkins' proof to establish duality between bases for forms with levels 2, 3, 4, 5, 7, 8, 9, 16, and 25, of every even weight

2017

Paul Jenkins and the author published **Zagier duality for level p weakly holomorphic modular forms**

They proved that duality holds for between weight 0 and weight 2 forms for an infinite class of primes, and that duality holds between weight k and $2-k$ forms for every prime ≤ 37 of nonzero genus

A few definitions

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Define $\widehat{M}_k^{(\infty)}(\Gamma, \nu)$ to be the space of weakly holomorphic modular forms with poles only at ∞ which vanish at each other cusp

Main theorem

Write $\left\{ f_{k,m}^{(\nu)}(z) = q^{-m} + \sum_{n} a_k^{(\nu)}(m,n) q^n \right\}_m$ for the reduced-echelon basis for $M_k^{(\infty)}(\Gamma, \nu)$

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Write $\left\{ g_{k,m}^{(\nu)}(z) = q^{-m} + \sum_n b_k^{(\nu)}(m,n) q^n \right\}_m$ for the reduced-echelon basis for $\widehat{M}_k^{(\infty)}(\Gamma, \nu)$

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Theorem (Griffin-Jenkins-M.)

$$a_k^{(\nu)}(m,n) = -b_{2-k}^{(\overline{\nu})}(n,m)$$

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Corollary (Griffin-Jenkins-M.)

$$\mathcal{F}_k^{(\nu)}(z,\tau) = -\mathcal{G}_{2-k}^{(\overline{\nu})}(\tau,z)$$

An example

The first few basis elements $f_{2,m}^{(11)}$ of $M_2^{(\infty)}(\Gamma_0(11))$ are:

$$
\begin{aligned}
f_{2,-1}^{(11)}(z) &= q &&- 2q^2 &&- q^3 &&+ 2q^4 &&+ q^5 + \dots \\
f_{2,0}^{(11)}(z) &= 1 &&+ 12q^2 &&+ 12q^3 &&+ 12q^4 &&+ 12q^5 + \dots \\
f_{2,1}^{(11)}(z) &= q^{-1} &&- 5q^2 &&- 2q^3 &&- 6q^4 &&+ 14q^5 + \dots \\
f_{2,2}^{(11)}(z) &= q^{-2} &&- 8q^2 &&- 2q^3 &&- 3q^4 &&+ 16q^5 + \dots
\end{aligned}
$$

The first few basis elements $g_{0,m}^{(11)}$ of $\widehat{M}_0^{(\infty)}(\Gamma_0(11))$ are:

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\begin{aligned}
g_{0,2}^{(11)}(z) &= q^{-2} &&+ 2q^{-1} &&- 12 &&+ 5q &&+ 8q^2 + \dots \\
g_{0,3}^{(11)}(z) &= q^{-3} &&+ 1q^{-1} &&- 12 &&+ 2q &&+ 2q^2 + \dots \\
g_{0,4}^{(11)}(z) &= q^{-4} &&- 2q^{-1} &&- 12 &&+ 6q &&+ 3q^2 + \dots \\
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$$f_{2,-1}^{(11)}(z) = q - 2q^2 - q^3 + 2q^4 + q^5 + \dots$$
$$f_{2,0}^{(11)}(z) = 1 + 12q^2 + 12q^3 + 12q^4 + 12q^5 + \dots$$
$$f_{2,1}^{(11)}(z) = q^{-1} - 5q^2 - 2q^3 - 6q^4 + 14q^5 + \dots$$
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A modest extension

Let $U \sqcup V \sqcup \{ \infty \}$ be a partition of the set of cusps $\Omega(\Gamma)$

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Define $\widehat{M}_k^{(\infty)}(\Gamma, \nu, U)$ to be the space of weakly holomorphic modular forms with poles only at ∞ which vanish on V

A modest extension

Write $\left\{ f^U_{\nu,k,m}(z) = q^{-m} + \sum_n a^U_{\nu,k}(m,n)q^n \right\}_m$ for the reduced-echelon basis for $M^\infty_k(\Gamma, \nu, U)$

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Theorem (Griffin-Jenkins-M.)

$$a^U_{\nu,k}(m,n) = -b^U_{\overline{\nu},2-k}(n,m)$$

Some notation

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Write ω_{ρ} for the cuspidal width of ρ

Some notation

We say $\mathbf{f} = \left(\mathbf{f}^{\lambda}\right)_{\lambda} \in \mathbb{C}((q))_{\Gamma,\nu}$ if

- Each $\mathbf{f}^{\lambda}$ is a formal Laurent series in q
- When $\lambda\infty = \lambda'\infty$, $\mathbf{f}^{\lambda}$ and $\mathbf{f}^{\lambda'}$ are compatible

$M_k^!(\Gamma,\nu) \hookrightarrow \mathbb{C}((q))_{\Gamma,\nu}$ via $f \mapsto (f|_k\lambda)_{\lambda}$

Write $\mathbf{f}^{\lambda} = \sum_n a^{\lambda}(n)q^n$, and $\mathbf{g}^{\lambda} = \sum_n b^{\lambda}(n)q^n$

Write ω_ρ for the cuspidal width of ρ

Choose γ_ρ so that $\gamma_\rho\infty = \rho$

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$$\{\bullet, \bullet\}_\Gamma : \mathbb{C}((q))_{\Gamma,\nu} \times \mathbb{C}((q))_{\Gamma,\overline{\nu}} \to \mathbb{C}$$

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Theorem (Bruinier-Funke)

If $f \in M^!_k(\Gamma,\nu)$ and $g \in M^!_{2-k}(\Gamma,\nu)$ then

$$\{\, f, g \,\}_\Gamma = 0$$

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Theorem (Borcherds)

For $\mathbf{f} = \left(\mathbf{f}^\lambda\right)_\lambda \in \mathbb{C}((q))_{\Gamma,\nu}$, TFAE:

- There exists $f \in M_k^!(\Gamma,\nu)$ such that for each λ, we have that $f^\lambda = \mathbf{f}^\lambda + o(1)$
- For every holomorphic modular form $g \in M_{2-k}(\Gamma,\overline{\nu})$, we have $\{\mathbf{f}, g\}_\Gamma = 0$

Proof of main theorem

Proof Sketch

$\left\{ f_{k,m}^{(\nu)}, g_{2-k,n}^{(\overline{\nu})} \right\} = 0$ as both forms are weakly holomorphic

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What about harmonic Maass forms?

Thank you for your attention!