On the classification of rigid meromorphic cocycles

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The setup

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- The values of these cocycles at RM points were studied by Darmon and Vonk in the paper *Singular moduli for real quadratic fields: a rigid analytic approach*.

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Conjecture (Darmon, Vonk)

$J[\tau]$ is an algebraic number in $\mathrm{H}_\tau \cdot \mathrm{H}_J$, where H_τ is the narrow ring class field associated to $\mathcal{O}_\tau$ and H_J is the compositum of the fields H_τ for $j(\tau) = \infty$.

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- Hence we first compute $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_2)$.
- Inspiration comes from the classification of $\mathrm{H}^1_{\mathrm{par}}(\mathrm{PSL}_2(\mathbb{Z}), M)$, where M are rational functions (Choie, Zagier).

The classifcation of rational period functions

- A *rational period function* (RPF) for $\mathrm{PSL}_2(\mathbb{Z})$ is a rational function q such that $q|_2(1+T) = 0 = q|_2(1+U+U^2)$, where T and U are the order 2 and 3 generators of $\mathrm{PSL}_2(\mathbb{Z})$.

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Theorem (Choie, Zagier)

Any RPF is a linear combination of the functions

$$\frac{1}{z} \text{ and } \phi_\tau(z) = \sum \mathrm{sgn}(\omega)\frac{1}{z-\omega},$$

where $\omega \in \mathrm{PSL}_2(\mathbb{Z})\tau$ for τ ranging through $\mathrm{PSL}_2(\mathbb{Z})$-representatives of simple real quadratic irrationalities.

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- Using the logarithmic derivative one gets:

Theorem (Darmon, Vonk)

For all primes p, the group $H^1_f(\Gamma, \mathcal{M}^\times)$ is of infinite rank over $\mathbb{Z}$.

Generalization

- We can ask what happens if Δ is a congruence subgroup of Γ, for example $\Delta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,\ c = 0 \text{ (mod q)},\ q \neq p \text{ prime} \right\}$.

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- Moreover, the sum used to define $\psi_\tau(z)$ does not converge anymore (the intersection of $\Delta\tau$ and any affinoid does not have the same number of positive and negative elements).
- A possible source of inspiration might be the work of Ash, who classified $\mathrm{H}^1_{\mathrm{par}}(G, M)$, where G is any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Thank you!