Intersection Numbers of Modular Geodesics

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Introduction

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- We would like to study the subset of closed geodesics.

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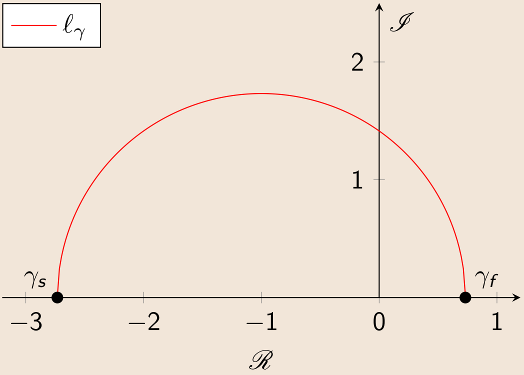
- All closed geodesics of $\Gamma\backslash\mathbb{H}$ arise in this way.

Example

$$\Gamma = \mathrm{PSL}(2,\mathbb{Z}), \qquad \gamma = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right), \qquad z = \frac{-5+\sqrt{3}i}{2}.$$



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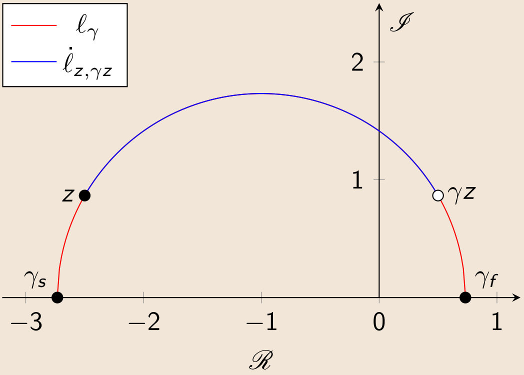

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- We will consider the intersections of $\tilde{\ell}_{\gamma_1}$ with $\tilde{\ell}_{\gamma_2}$ for γ_1, γ_2 strongly inequivalent.

Intersection number definition

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Let f be a function, let γ_1, γ_2 be strongly inequivalent, and define the $f-$weighted intersection number to be

$$\mathrm{Int}_{\Gamma}^{f}(\gamma_1, \gamma_2) = \sum_{\tilde{z} \in \tilde{\ell}_{\gamma_1} \cap \tilde{\ell}_{\gamma_2}} f(\tilde{z}, \tilde{\ell}_{\gamma_1}, \tilde{\ell}_{\gamma_2}).$$

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- When $\Gamma \backslash \mathbb{H}$ is a Shimura curve, we can define a $p-$weighted intersection, $\mathrm{Int}_{\Gamma}^{p}$, for primes p.

Alternate interpretation of the intersection number

- Define an equivalence relation on $\Gamma \times \Gamma$ by simultaneous conjugation, i.e.

$$(\sigma_1, \sigma_2) \sim (\alpha\sigma_1\alpha^{-1}, \alpha\sigma_2\alpha^{-1}) \text{ for } \alpha \in \Gamma.$$

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- This lifting is unique up to $\sim$. In particular,

$$\mathrm{Int}_\Gamma^f(\gamma_1, \gamma_2) = \sum_{\substack{(\sigma_1,\sigma_2)\in([\gamma_1]\times[\gamma_2])/\sim \\ \ell_{\sigma_1}\cap\ell_{\sigma_2}\neq\emptyset}} f(\sigma_1, \sigma_2).$$

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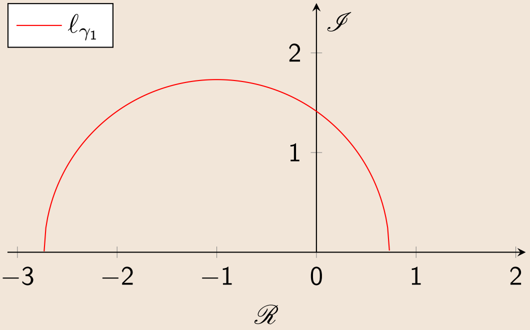

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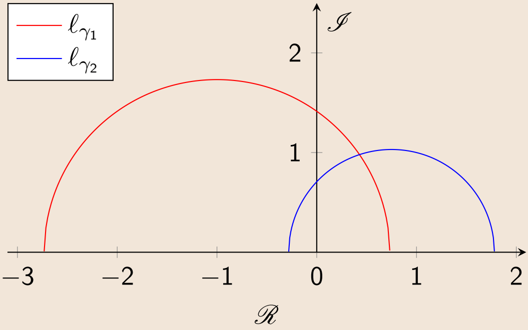

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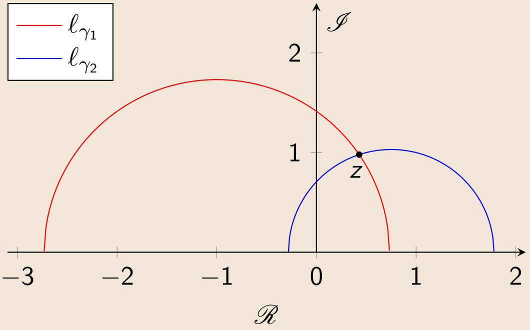

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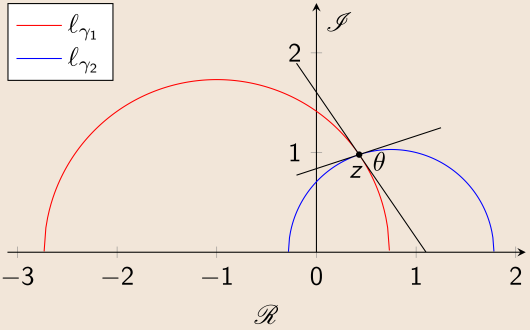

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Optimal embeddings in Eichler orders

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- Let D be a discriminant, and let $\mathcal{O}_D$ be the unique quadratic order of discriminant D. An optimal embedding of $\mathcal{O}_D$ into $\mathbb{O}$ is a ring homomorphism $\phi : \mathcal{O}_D \to \mathbb{O}$ which does not extend to an embedding of a larger quadratic order.

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- Let ϵ_D be the fundamental unit of positive norm in $\mathcal{O}_D$.
- Note that $\iota(\phi(\epsilon_D))$ is a primitive hyperbolic matrix in Γ, and all such matrices arise in this fashion.

Intersection number of optimal embeddings

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- When $\Gamma = \mathrm{PSL}(2, Z)$, there is a canonical basepoint, and we can replace "equivalence class of optimal embedding" with "primitive indefinite binary quadratic form".

Intersection point and angle

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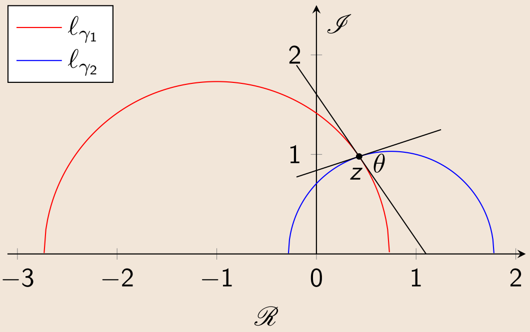
- If the root geodesics intersect, the intersection angle θ satisfies
$$\tan(\theta) = \frac{\sqrt{D_1 D_2 - x^2}}{x}.$$

Example, revisited



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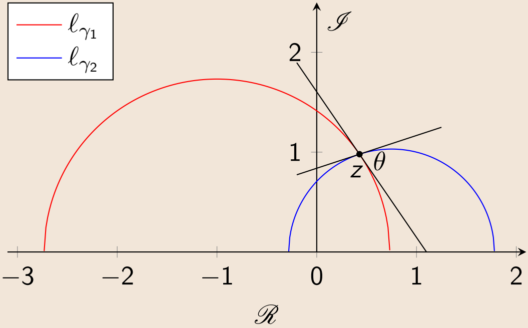

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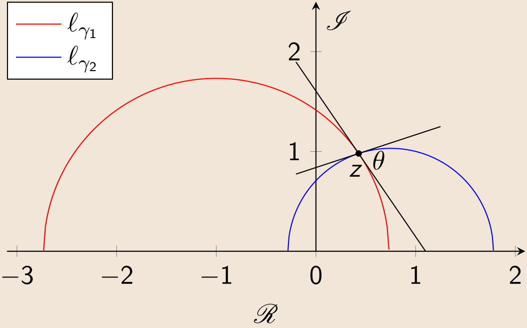

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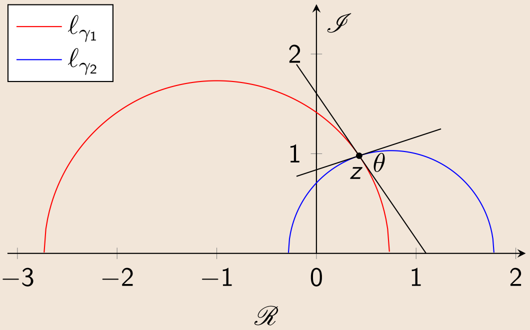

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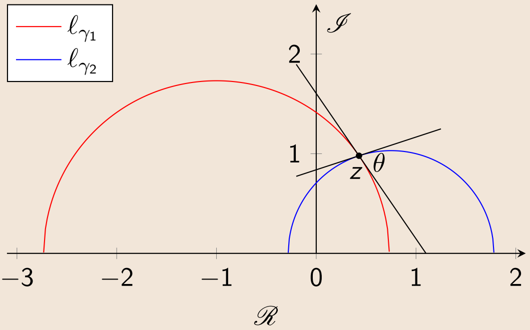

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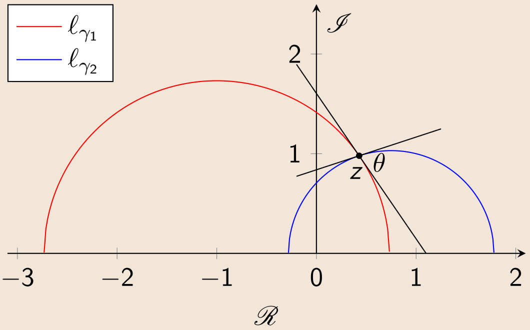

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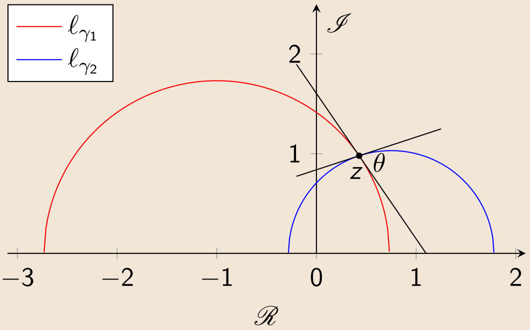
$\phi_1(\sqrt{12})\phi_2(\sqrt{17}) = \begin{pmatrix} 10 & -16 \\ 14 & -2 \end{pmatrix}$
$x = 4$
$7z^2 - 6z + 8 = 0$

Example, revisited



$\Gamma = \mathrm{PSL}(2, \mathbb{Z})$

$\gamma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

$\gamma_2 = \begin{pmatrix} 57 & 16 \\ 32 & 9 \end{pmatrix}$

$z = \dfrac{3 + \sqrt{47}i}{7}$

$\theta \approx 1.28694 \approx 73.7362^\circ$

$D_1 = 12$
$D_2 = 17$
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- All implementations are done in GP/PARI.

Definition of ϵ

- For simplicity, assume that D_1, D_2 are positive coprime fundamental discriminants. For all primes p with $\left(\frac{D_1 D_2}{p}\right) \neq -1$, define

$$\epsilon(p) := \begin{cases} \left(\dfrac{D_1}{p}\right) & \text{if } p \text{ and } D_1 \text{ are coprime;} \\ \left(\dfrac{D_2}{p}\right) & \text{if } p \text{ and } D_2 \text{ are coprime.} \end{cases}$$

Extend this multiplicatively.

- If $x \equiv D_1 D_2 \pmod{2}$, then

$$\epsilon\left(\frac{D_1 D_2 - x^2}{4}\right) = 1.$$

Existence of intersection

Theorem

Let B be the indefinite quaternion algebra of discriminant $\mathfrak{D}$, and let $\mathbb{O}$ be a maximal order. Factorize

$$\frac{D_1 D_2 - x^2}{4} = \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where $\epsilon(p_i) = \epsilon(q_i) = -\epsilon(w_i) = -1$. Then,

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- *There exist $x-$linked optimal embeddings of discriminants D_1, D_2 if and only if $\mathfrak{D} = \prod_{i=1}^{r} p_i$.*
- *The number of pairs of $x-$linked optimal embeddings of discriminants D_1, D_2 up to simultaneous conjugation is equal to*

$$2^{r+1}\prod_{i=1}^{t}(g_i+1) = 2^{r+1}\sum_{d \mid \frac{D_1D_2-x^2}{4\mathfrak{D}}} \epsilon(d).$$

Consequences

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- Each such x corresponds to a unique quaternion algebras for which there are intersections.
- Therefore, for all but finitely many quaternion algebras, the modular geodesics corresponding to optimal embeddings of discriminants D_1, D_2 will not intersect.

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- Let $q = e^{2\pi i\theta}$, and form the formal power series

$$E_{\phi_1,\phi_2}(\theta) := \sum_{n=1}^{\infty} \langle [\phi_1], T_n[\phi_2] \rangle q^n.$$

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$$E_{\phi_1,\phi_2}(\theta) \in S_2(\mathfrak{D}\mathfrak{M}).$$

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- In “Modular cocycles and linking numbers” by Duke, Imamoḡlu, and Tóth, the linking numbers of certain links on the space $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$ are considered. These linking numbers correspond exactly to unweighted intersection numbers in case $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

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- Let τ_1, τ_2 be real quadratic, representing coprime fundamental discriminants, and let p be a prime. In "Singular moduli for real quadratic fields", Darmon and Vonk derive generate a $p-$adic number $J_p(\tau_1, \tau_2)$, which is conjecturally algebraic and a real quadratic analgoue to $j(\tau_1) - j(\tau_2)$. The valuations of J_p at primes lying above q are conjectured to be $q-$weighted intersection numbers. This conjecture has been computationally verified for a large amount of data.

Acknowledgments and References

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