Counting elliptic curves with an isogeny of degree three

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Torsion is rare

To quantify the fact that most elliptic curves do not have torsion, we count as follows.

Every elliptic curve E over $\mathbb{Q}$ is uniquely of the form

$$E\colon y^2 = x^3 + Ax + B$$

with $A, B \in \mathbb{Z}$ satisfying $4A^3 + 27B^2 \neq 0$ and such that there is no prime ℓ such that $\ell^4 \mid A$ and $\ell^6 \mid B$.

For each such elliptic curve E, define its *height*:

$$\operatorname{ht} E := \max(\left|4A^3\right|, \left|27B^2\right|).$$

By Mazur's theorem, there are only finitely many possible groups G such that $E(\mathbb{Q})_{\mathrm{tors}} \simeq G$. For each such group G, Harron–Snowden prove that

$$N_G(H) := \#\{E : \operatorname{ht}(E) \leq H \text{ and } E(\mathbb{Q})_{\mathrm{tors}} \simeq G\} \asymp H^{1/d(G)}$$

for H large, where $d(G)$ is given explicitly.

Torsion is rare (Harron–Snowden)

$$N_G(H) := \#\{E : \mathrm{ht}(E) \leq H \text{ and } E(\mathbb{Q})_{\mathrm{tors}} \simeq G\} \asymp H^{1/d(G)}$$

G	# curves $= H^{1/d(G)}$
–	$H^{5/6}$
$\mathbb{Z}/2\mathbb{Z}$	$H^{1/2}$
$\mathbb{Z}/3\mathbb{Z}$	$H^{1/3}$
$\mathbb{Z}/4\mathbb{Z}$	$H^{1/4}$
$\mathbb{Z}/5\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/6\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/7\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/8\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/9\mathbb{Z}$	$H^{1/18}$
$\mathbb{Z}/10\mathbb{Z}$	$H^{1/18}$
$\mathbb{Z}/12\mathbb{Z}$	$H^{1/24}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$H^{1/3}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	$H^{1/24}$

Explicit asymptotics

To study $\#\{E : \mathrm{ht}(E) \leq H\}$, we need to count pairs

$$(A, B) \in \mathbb{Z}^2 \text{ such that } |A| \leq (H/4)^{1/3}, |B| \leq (H/27)^{1/2}$$

and then sieve out those with $\ell^4 \mid A, \ell^6 \mid B$ for some prime ℓ; the number with $4A^3 + 27B^2 = 0$ are only $O(H^{1/6})$. So we need to count lattice points in a rectangle with sides of lengths $2(H/4)^{1/3}$ and $2(H/27)^{1/2}$, as $H \to \infty$.

By the Principle of Lipschitz, the number of lattice points in a region is given by its area up to an error proportional to length of its (rectifiable) boundary. So the above count is

$$4(1/4)^{1/3}(1/27)^{1/2}H^{5/6} + O(H^{1/2}).$$

The condition at ℓ says we have *overcounted* and need to multiply the result by $(1 - \ell^{10})$; a standard sieve argument then gives

$$\#\{E : \mathrm{ht}(E) \leq H\} = 2^{4/3}3^{-3/2}\zeta(10)^{-1}H^{5/6} + O(H^{1/2}).$$

Explicit asymptotics: $\#G = 2, 3$

Harron–Snowden carried out this strategy for the cases $\#G = 2, 3$:

$$\#\{E : \mathrm{ht}(E) \leq H \text{ and } E(\mathbb{Q})_{\mathrm{tors}} \simeq G\} \\ = \frac{\mathrm{area}(R_G)}{\zeta(12/d(G))} H^{1/d(G)} + O(H^{1/e(G)})$$

for

G	$H^{1/d(G)}$	$O(H^{1/e(G)})$
–	$H^{5/6}$	$O(H^{1/2})$
$\mathbb{Z}/2\mathbb{Z}$	$H^{1/2}$	$O(H^{1/3})$
$\mathbb{Z}/3\mathbb{Z}$	$H^{1/3}$	$O(H^{1/4})$

Question

Without computing the constant, can one use this method to prove there exists an effectively computable constant for all G?

Counting isogenies

In this talk, we are concerned not with counting elliptic curves with torsion subgroups but rather elliptic curves *with an isogeny (over $\mathbb{Q}$)*.

For $m \in \mathbb{Z}_{\geq 1}$, let

$$N_m(H) := \left\{ E : \begin{array}{l} \mathrm{ht}(E) \leq H \text{ and} \\ \text{there exists } \phi\colon E \to E' \text{ cyclic of degree } m \end{array} \right\}.$$

For $m = 1, 2$, a generator of the kernel of a cyclic m-isogeny is a rational m-torsion point, so we are in the previous case: $N_m(H) = N_{\mathbb{Z}/m\mathbb{Z}}(H)$ for $m = 1, 2$. (In terms of modular curves, $X_1(m) = X_0(m)$ for $m = 1, 2$.)

Main result: counting cyclic 3-isogenies

Theorem (Pizzo–Pomerance–V)

There exist $c_1, c_2 \in \mathbb{R}$ *such that for* $H \geq 1$,

$$N_3(H) = \frac{2}{3\sqrt{3}\zeta(6)} H^{1/2} + c_1 H^{1/3} \log H + c_2 H^{1/3} + O(H^{7/24}).$$

Moreover,

$$\frac{2}{3\sqrt{3}\zeta(6)} = 0.378338\ldots \qquad c_1 = \frac{c_0}{8\pi^2\zeta(4)} = 0.107437\ldots$$

where c_0 *is explicitly given and* c_2 *is effectively computable.*

- Same asymptotic if we count those *equipped* with a 3-isogeny.
- The main term of order $H^{1/2}$ counts *just* those elliptic curves with $A = 0$ (having j-invariant 0).
- Matches computations to $H = 10^{25}$, suggesting $c_2 \approx 0.163$.
- 3-isogenies are more frequent than $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-torsion, by a log factor (its first appearance).

A hint of the proof

An elliptic curve E has a 3-isogeny (defined over $\mathbb{Q}$) if and only if its 3-division polynomial

$$\psi(x) = 3x^4 + 6Ax^2 + 12Bx - A^2$$

has a root in $\mathbb{Q}$; if $a \in \mathbb{Q}$ is such a root, then in fact $a \in \mathbb{Z}$.

So we need to count triples $(A, B, a) \in \mathbb{Z}^3$ satisfying:

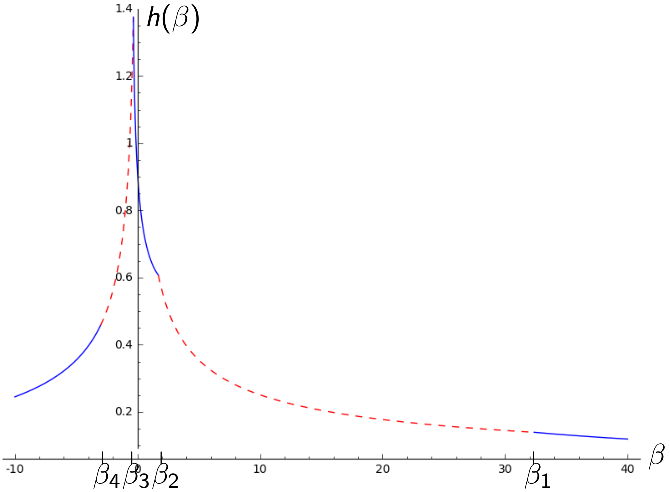
(N1) $A \neq 0$ and $\psi_{A,B}(a) = 0$;
(N2) $\left|4A^3\right|, \left|27B^2\right| \leq H$;
(N3) $4A^3 + 27B^2 \neq 0$; and
(N4) there is no prime ℓ with $\ell^4 \mid A$ and $\ell^6 \mid B$.

We show that this count is of size $H^{1/3} \log H$; we use that

$$12B = \frac{A^2}{a} - 6Aa - 3a^3$$

so it is enough to work with A, a such that $a \mid A^2$, together with conditions at $2, 3$.

Our region



$h(\beta)$
1.4
1.2
1
0.8
0.6
0.4
0.2
-10
0
10
20
30
40
β_4 β_3 β_2
β_1
β

Geometric interpretation; conclusion

$N_3(H)$ counts rational points of bounded height on $X_0(3)$ with respect to the height arising from the pullback of the natural height on $X(1)$.

The main term corresponds to a single elliptic point of order 3 on $X_0(3)$! The modular curves $X_0(N)$ are not fine moduli spaces (owing to quadratic twists), so our proof is quite different than the method used by Harron–Snowden.

We hope that our method and the lower-order terms in our result will be useful in understanding counts of rational points on stacky curves (as in recent work of Ellenberg–Satriano–Zureick-Brown).

Question

Can one predict the answer for the count of elliptic curves with a cyclic m-isogeny for general m (in terms of the signature of $X_0(m)$?)?