On the total number of prime factors of an odd perfect number

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- Is there an odd N such that $\sigma(N) = 2N$?
- Throughout this talk N will denote an odd perfect number.

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Theorem

(Z.) If N is an odd perfect number with $3 \nmid N$, then

$$\Omega \geq \frac{302}{113}\omega - \frac{286}{133}. \tag{1}$$

If N is an odd perfect number, with $3|N$, then

$$\Omega \geq \frac{66}{25}\omega - 5. \tag{2}$$

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- Euler's result follows immediately from considering what happens mod 4.
- $\sigma(p^2) = p^2 + p + 1$, and $\sigma(p^4) = p^4 + p^3 + p^2 + p + 1$.
- If $p|n^2 + n + 1$, then $p \equiv 1 \pmod 3$ or $p = 3$. Similar statement for $p|n^4 + n^3 + n^2 + n + 1$.

Central ingredients to Ochem and Rao

- Key insight: Either we have many copies of 3 in the factorization, or we have many primes raised to a power greater than 2.
- Use a system of linear inequalities on the number of prime factors.
- If $p \equiv 1 \pmod{3}$, then $3|\sigma(p^2)$.

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- Key insight: Either we have many copies of 3 in the factorization, or we have many primes raised to a power greater than 2.
- Use a system of linear inequalities on the number of prime factors.
- If $p \equiv 1 \pmod 3$, then $3|\sigma(p^2)$.
- If $p^2||N$, and $q|\sigma(p^2)$, then either $q^4|N$ or q contributes a 3.

Lemma (Ochem and Rao)

Let p, q and r be positive integers. If $p^2 + p + 1 = r$ and $q^2 + q + 1 = 3r$, then p is not an odd prime.

- Look at number of primes in S (set of primes of N which are raised to the second power), and the number of primes in T (set of primes which are raised to the fourth power), and U set of primes raised to higher powers. Keep the special prime and the powers of 3 separate.

Additional ingredients

Lemma

Let a and b be distinct odd primes and p a prime such that $p|(a^2 + a + 1)$ and $p|(b^2 + b + 1)$. If $a \equiv b \equiv 2$ (mod 3), then $p \leq \frac{a+b+1}{5}$. If $a \equiv b \equiv 1$ (mod 3), then $p \leq \frac{a+b+1}{3}$.

Forces a large set of distinct primes from S.

Major obstruction

- Define a Triple Threat as a quadruplet of primes (x, a, b, c) with $\sigma(a^2)$, $\sigma(b^2)$ and $\sigma(c^2)$ also prime and

$$\sigma(x^2) = \sigma(a^2)\sigma(b^2)\sigma(c^2).$$

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- Combine this with many primes in S being 1 (mod 5).
- Primes in T contribute a lot or we have a lot of info about the primes: Set $g(x) = x^4 + x^3 + x^2 + x + 1$, $f(x) = x^2 + x + 1$. Then $f(g(x)) = (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3)$.

Did we get lucky?

Conjecture

Let p and q be distinct odd primes and let $\Phi_p(x)$ and $\Phi_q(x)$ be the pth and qth cyclotomic polynomials. Then aside from a finite set of exceptions, at least one of $\Phi_p(\Phi_q(x))$ or $\Phi_q(\Phi_p(x))$ is irreducible.

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- Call an ordered pair of positive integers (m, n) a *good pair* if $\Phi_m(\Phi_n(x))$ factors over the integers where Φ_m and Φ_n are the mth and nth cyclotomic polynomials. Let $D(t)$ count the number of good pairs with both $m \leq t$ and $n \leq t$.

Conjecture

$$\lim_{t \to \infty} \frac{D(t)}{t^2} = 0.$$

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- Understanding cyclotomic polynomial composition.
- Generalizing these results (Multiply perfect numbers, Ore harmonic numbers).
- Can we get a better than linear inequality?

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