Bessel functions outside $GL(2)$.

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What are Bessel functions outside of $GL(2)$?

Let

- $G = PGL(n, \mathbb{R}) = GL(n, \mathbb{R})/\mathbb{R}^\times$,
- $U(\mathbb{R})$ the upper triangular unipotent matrices,
- Y the diagonal matrices,
- Y^+ the positive diagonal matrices,
- $K = PO(n, \mathbb{R})$.

What are Bessel functions outside of $GL(2)$?

Define characters

- of Y:

$$p_\mu\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n |a_i|^{\mu_i}, \qquad \chi_\delta\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n \operatorname{sgn}(a_i)^{\delta_i},$$

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- of $U(\mathbb{R})$: $\psi_y(x) = \psi_I(yxy^{-1}) = e\left(y_1x_1 + \ldots + y_{n-1}x_{n-1}\right)$,

$$y = \begin{pmatrix} y_1\cdots y_{n-1} & & & & \\ & y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 & x_n & & * \\ & \ddots & \ddots & \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix},$$

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- normalizations: $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}\right)$, use $p_{\rho+\mu}$, assume $\sum_{i=1}^{n} \mu_i = 0$, extend to G by $p_{\rho+\mu}(xyk) = p_{\rho+\mu}(y)$.

Spherical Jacquet-Whittaker function:

$$W(g,\mu,\psi) = \int_{U(\mathbb{R})} p_{\rho+\mu}(w_l x g)\overline{\psi(x)}dx, \qquad w_l = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

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Since $p_{\rho+\mu}$ is an eigenfunction of the Casimir operators, so is the Whittaker function, with the same eigenvalues.

Call μ the (Harish-Chandra/Langlands) spectral parameters.

What are Bessel functions outside of $GL(2)$?

Conjecture (Interchange of Integrals)

If $f(\mu)$ is homorphic with rapid decay on an open tube domain containing $\text{Re}(\mu) = 0$, and $y \in Y, t \in G$, then

$$\int_{U(\mathbb{R})} \int_{\text{Re}(\mu)=0} f(\mu) W(yw_l xt, \mu, \psi_I) d\mu \, \overline{\psi_I(x)} dx$$
$$= \int_{\text{Re}(\mu)=0} f(\mu) \widetilde{K}_{w_l}(y, t, \mu) d\mu,$$

where

$$\widetilde{K}_{w_l}(y, t, \mu) := \lim_{R \to \infty} \int_{U(\mathbb{R})} h\left(\frac{\|x\|}{R}\right) W(yw_l xt, \mu, \psi_I) \overline{\psi_I(x)} dx$$

is smooth in t and y, entire in μ and polynomially bounded in the coordinates of $y, t, y^{-1}, t^{-1}, \mu$ for $\text{Re}(\mu)$ in some fixed, compact set and h smooth and compactly supported with $h(0) = 1$.

Convergence!

What are Bessel functions outside of $GL(2)$?

It follows from work of Shalika that

$$\widetilde{K}_{w_l}(y, t, \mu) = K_{w_l}(y, \mu) W(t, \mu, \psi_I)$$

for some function $K_{w_l}(y, \mu)$, and this is called the long-element, spherical Bessel function for $GL(n)$.

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$$K_{w_l}(xy,\mu) = K_{w_l}(y(w_l x w_l),\mu) = \psi_I(x) K_{w_l}(y,\mu),$$

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There are Bessel functions $K_w(y, \mu, \delta)$ attached to the other elements of the Weyl group and to non-spherical $\chi_\delta \neq 1$, as well.

What do we know about these Bessel functions?

Conjecture (Asymptotics)

Dropping the μ integral and ignoring issues of convergence, replacing the Whittaker function in the definition of $K_w(y, \mu, \delta)$ with its first-term asymptotics as $Y \ni y \to 0$ yields the first-term asymptotics of $K_w(y, \mu, \delta)$.

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Conjecture (Uniqueness)

The function $K_w(y, \mu, \delta)$ is uniquely determined by its first-term asymptotics, bi-$U(\mathbb{R})$-invariance and eigenvalues under the Casimir operators.

What do we know about these Bessel functions?

The uniqueness conjecture is true in the long element case:

- $p_{-\rho}(g)K_{w_l}(gg^T, \mu, \delta)$ is the spherical Whittaker function $W(g, 2\mu, \psi_I^2)$, up to a function $C(\mu, \delta)$

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- from Hashizume's solution of the Whittaker differential equations:

$$K_{w_l}(y, \mu, \delta) = \sum_{w \in W} C_w(\mu, \delta, \operatorname{sgn}(y)) J_{w_l}(y, \mu^w),$$

$$J_{w_l}(y, \mu) = p_{\rho+\mu}(y) \sum_{m \in \mathbb{N}_0^{n-1}} a_m(\mu)(4\pi^2 y)^m$$

$$\left(\sum_{j=0}^{n-1}(m_j - m_{j+1})^2 - \sum_{j=1}^{n-1} m_{n-j}(\mu_{j+1} - \mu_j)\right) a_m(\mu) = \sum_{j=1}^{n-1} a_{m-e_j}(\mu)$$

$$a_0(\mu) = 1, \qquad m_0 = m_n = 0.$$

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- The asymptotics determine $C_w(\mu, \delta, \operatorname{sgn}(y))$.

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- The trivial Bessel function is 1; the Voronoi Bessel function (probably) also shows up.

Work in progress.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

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Theorem (B)

Assuming the interchange of integrals, the asymptotics conjecture is true for all Weyl elements in the spherical and non-spherical cases on $GL(n)$.

Applications - Spectral Kuznetsov trace formula.

Take the Fourier coefficient of a Poincaré series using Wallach's Whittaker inversion formula and Langland's spectral expansion:

$$\int_{\mathcal{B}(\sigma)} \rho_\xi(n)\overline{\rho_\xi(m)} f(\mu_\xi) W(t, \mu_\xi, \chi, \sigma, \psi_I) d\xi$$
$$= \sum_{w \in W} \sum_{c \in A(\mathbb{Z})} p_\rho(c) S_w(\psi_m, \psi_n, c) H_w(f; mcwn^{-1}w^{-1}),$$

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General Stone-Weierstrass-type argument to get rid of the spare Whittaker function?
Still needs Stade's formula to convert to Hecke eigenvalues.

From the Spectral Kuznetsov formula, the Mellin-Barnes integrals of the Bessel functions and good bounds for the Kloosterman sums, we get

$$\int_{\substack{\mathcal{B}(\sigma) \\ \mu_\xi \in \Omega}} \frac{1}{L(1, \mathrm{Ad}^2 \xi)} d\xi = \int_\Omega d_{\mathrm{spec}}\mu + O\left(\int_{\partial\Omega + B(0,1)} d_{\mathrm{spec}}\mu\right)$$

Theorem (Blomer/B)

Arithmetically-weighted Weyl laws with error term for $SL(3, \mathbb{Z})$.

On a set $T\Omega$, we can probably improve the radius in the error term from 1 to $(\log T)^{-\delta}$ for some $\delta \in (0, \frac{1}{2})$.

With much work, we get

Theorem (Blomer,B)

For ϕ a cusp form for $SL(3,\mathbb{Z})$ such that μ_ϕ is in "generic position",

1. If ϕ is spherical, then $L(\frac{1}{2},\phi) \ll \|\mu_\phi\|^{\frac{3}{4}-\frac{1}{120000}}$.
2. If ϕ has weight $d \geqslant 3$, then $L(\frac{1}{2},\phi) \ll \|\mu_\phi\|^{\frac{3}{4}-\frac{1}{140000}}$.

Generic position: $\mu_i, \mu_i - \mu_j \asymp \|\mu\|$.

Thank you!