Computing with (indefinite) quadratic forms and quaternion algebras in PARI/GP

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- If $\gamma \in \Gamma$ is primitive and hyperbolic, its *root geodesic* is the upper half plane geodesic connecting the two (real) roots.
- This descends to a closed geodesic in $\Gamma\backslash\mathbb{H}$, and all closed geodesics arise in this fashion.

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- The case of $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ relates to the work of Duke, Imamoḡlu, and Tóth on linking numbers of modular knots in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ ([DIT17]).

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- The Shimura curve case (conjecturally) relates to the work of Darmon and Vonk on real quadratic analogues of the $j-$function ([DV17]).
- There are lots of parallels to the work of Gross and Zagier on the factorization of the difference of $j-$values ([GZ85]).

The setup for $\mathrm{PSL}(2, \mathbb{Z})$

- Let $q(x, y)$ be a primitive indefinite binary quadratic form (PIBQF), let γ_q be its automorph, and let ℓ_q be the geodesic connecting the roots of q.

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- This translates the inputs into pairs of PIBQFs, which come equipped with discriminants.
- In fact, we can descend to equivalence classes of PIBQFs, since the root geodesic in $\Gamma\backslash\mathbb{H}$ does not depend on the representative.

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- Let B be an indefinite quaternion algebra over $\mathbb{Q}$, $\mathbb{O}$ an Eichler order in B, and $\iota : B \to \mathrm{Mat}_2(\mathbb{R})$ an embedding.

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- Then $\iota(\phi(\epsilon_D)) \in \Gamma$ is a hyperbolic element.

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- Then $\iota(\phi(\epsilon_D)) \in \Gamma$ is a hyperbolic element.
- Thus we take the inputs to be pairs of (equivalence classes of) optimal embeddings, which again come equipped with discriminants.

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- GP: a scripting language that allows “on the go” access to the tools in PARI.

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- PARI: a C library with an extensive amount of number theoretic tools.
- GP: a scripting language that allows “on the go” access to the tools in PARI.
- Initially, I was working exclusively in GP.

Finding interesting examples

Question

Given positive discriminants D_1, D_2, which quaternion algebras admit optimal embeddings into a maximal order that have non-trivial intersections?

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- Ran a bunch of computations on quaternion algebras ramifying at $\{p, q\}$, and produced a finite list of such pairs for each pair of discriminants.
- Possible ramifying primes were always "small", and missing certain primes, even when the discriminants grew.

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Given positive discriminants D_1, D_2, which quaternion algebras admit optimal embeddings into a maximal order that have non-trivial intersections?

- Turned this data into the conjecture

$$pq \mid \frac{D_1 D_2 - x^2}{4}$$

for some integer x with $x \equiv D_1 D_2 \pmod{2}$ and $|x| < \sqrt{D_1 D_2}$.

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- Computations were valuable to help verify the more precise conjecture in some of the messier cases.

Connection with Darmon-Vonk

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- The data matched perfectly!

Q-Quadratic Package

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- In addition, there are two users manuals: one for PARI and one for GP (currently they are 58 and 26 pages long respectively).
- I am uploading the package to my Github (live version is 0.3): `https://github.com/JamesRickards-Canada/Q-Quadratic`

Documentation excerpt

3.1 Discriminant methods

These methods deal with discriminant operations that do not involve quadratic forms.

Name:	`GEN disclist`
Input:	`GEN D1, GEN D2, int fund, GEN cop`
Input format:	Integers `D1`, `D2`, `fund=0, 1`, `cop` an integer
Output format:	Vector
Description:	Returns the set of discriminants (non-square integers equivalent to `0, 1` modulo 4) between `D1` and `D2` inclusive. If `fund=1`, only returns fundamental discriminants, and if `cop`$\neq$`0`, only returns discriminants coprime to `cop`.

Name:	`GEN discprimeindex`
Input:	`GEN D, GEN facs`
Input format:	Discriminant `D`, `facs=0` or the factorization of `D` (the output of `Z_factor`)
Output format:	Vector
Description:	Returns the set of primes p for which D/p^2 is a discriminant.

Name:	`GEN discprimeindex_typecheck`
Input:	`GEN D`
Input format:	Discriminant `D`
Output format:	Vector
Description:	Checks that `D` is a discriminant, and returns `discprimeindex(D, gen_0)`.

Name:	`GEN fdisc`
Input:	`GEN D`
Input format:	Discriminant `D`
Output format:	Integer
Description:	Returns the fundamental discriminant associated to `D`.

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Implemented algorithms

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- Computing the Conway rivers associated to a PIBQF, as well as left/right neighbours of reduced forms.
- Finding the general integer solution set to the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = n,$$

as well as the simultaneous equations

$$AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = n_1, \qquad GX + HY + IZ = n_2.$$

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- Initializing quaternion algebras, maximal orders, and doing all the basic operations.
- Computing all optimal embeddings, and sorting them by orientation and the class group action.
- Compute the intersection number via "intersecting root geodesics", as well as "x-linking".

Sample output

```
(12:21) gp > [Q, order]=qa_init_2primes(2, 7)
%1 = [[0, [2, 7], [7, -1, 7], 14], [[1, 0, 0, 1/2; 0, 1, 0, 1/2; 0, 0, 1, 1/2; 0, 0, 0, 1/2], 0, [2, 2, 2, 2], 1, [], [1
, 0, 0, -1; 0, 1, 0, -1; 0, 0, 1, -1; 0, 0, 0, 2], [[0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]]]]
(12:21) gp > d1s=qa_embeddablediscs(Q, order, 1, 200, 1, 2021)
%2 = [5, 12, 13, 21, 24, 28, 40, 56, 61, 69, 76, 77, 101, 104, 124, 133, 136, 140, 152, 157, 168, 173, 181]
(12:21) gp > e1s=qa_sortedembed(Q, order, 61)
%3 =
[    [] [[1/2, -1/2, 3/2, -3/2]]]

[   [2]  [[1/2, -1/2, 3/2, 3/2]]]

[   [7] [[1/2, 1/2, -3/2, -3/2]]]

[[2, 7]  [[1/2, 1/2, -3/2, 3/2]]]

(12:21) gp > e2s=qa_sortedembed(Q, order, 2021)
time = 47 ms.
%4 =
[[] [[1/2, -1/2, 3/2, -17/2], [1/2, -13/2, 3/2, 11/2], [1/2, -7/2, 45/2, 23/2], [1/2, -29/2, 87/2, 23/2], [1/2, -23/2, 4
5/2, 7/2], [1/2, -11/2, 3/2, 13/2]]]

[[2] [[1/2, -1/2, 3/2, 17/2], [1/2, -11/2, 3/2, -13/2], [1/2, -23/2, 45/2, -7/2], [1/2, -29/2, 87/2, -23/2], [1/2, -7/2,
 45/2, -23/2], [1/2, -13/2, 3/2, -11/2]]]

[[7] [[1/2, 29/2, -87/2, 23/2], [1/2, 23/2, -45/2, 7/2], [1/2, 11/2, -3/2, 13/2], [1/2, 1/2, -3/2, -17/2], [1/2, 13/2, -
3/2, 11/2], [1/2, 7/2, -45/2, 23/2]]]

[[2, 7] [[1/2, 29/2, -87/2, -23/2], [1/2, 7/2, -45/2, -23/2], [1/2, 13/2, -3/2, -11/2], [1/2, 1/2, -3/2, 17/2], [1/2, 11
/2, -3/2, -13/2], [1/2, 23/2, -45/2, -7/2]]]
```

Sample output

```
(12:21) gp > qa_inum_roots(Q, order, e1s[1,2][1], e2s[1,2][1])
%5 = [[[1/2, 3/2, 3/2, 1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, -5/2, -11/2, -1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, 7/2,
17/2, -1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, -15/2, -39/2, 1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, 17/2, 45/2, -3/2], [1
/2, -1/2, 3/2, -17/2]], [[1/2, -41/2, -109/2, 5/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, 73/2, -193/2, -1/2], [1/2, -1/2, 3/
2, -17/2]], [[1/2, -33/2, 87/2, 1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, 15/2, -39/2, -1/2], [1/2, -1/2, 3/2, -17/2]], [[1
/2, -7/2, 17/2, 1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, 5/2, -11/2, 1/2], [1/2, -1/2, 3/2, -17/2]], [[1/2, -3/2, 3/2, -1/
2], [1/2, -1/2, 3/2, -17/2]]]
(12:21) gp > length(%)
%6 = 12
(12:21) gp > qa_inum_x(Q, order, e1s[1,2][1], e2s[1,2][1])
time = 31 ms.
%7 = [[[1/2, -1/2, 3/2, -3/2], [1/2, 750167/2, -1986765/2, 33767/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, -3601/2, 9537/2, -1
63/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, -271/2, 717/2, -17/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, 17/2, 45/2, -17/2]], [[1/2
, -1/2, 3/2, -3/2], [1/2, 2425/2, -6423/2, 115/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, -985271/2, 2609421/2, -44345/2]], [[1
/2, -1/2, 3/2, -3/2], [1/2, 127/2, -333/2, -1/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, -17/2, 3/2, -1/2]], [[1/2, -1/2, 3/2,
-3/2], [1/2, -145/2, 381/2, 1/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, 17/2, 3/2, 1/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, 67290
5/2, -1782141/2, 30281/2]], [[1/2, -1/2, 3/2, -3/2], [1/2, -2887/2, 7647/2, -139/2]]]
(12:21) gp > length(%)
%8 = 12
(12:21) gp >
```

Planned algorithms

- Computing the fundamental domain for Shimura curves (partially working prototype in GP, not yet transferred over). See Voight [Voi09] and Page [Pag15] for the algorithms.

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- Use said algorithm to improve the computation of optimal embeddings.

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- Solve the principal ideal problem for indefinite quaternion algebras (algorithm due to Page [Pag14]).
- Use said algorithm to improve the computation of optimal embeddings.
- Continue to implement useful basic quaternion algebra methods.

Acknowledgments and References

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