The hybrid topology
A bridge between QFT and string theory

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Joint work with O. Amini, S. Bloch, J. Fresán

A typical experiment in high energy physics

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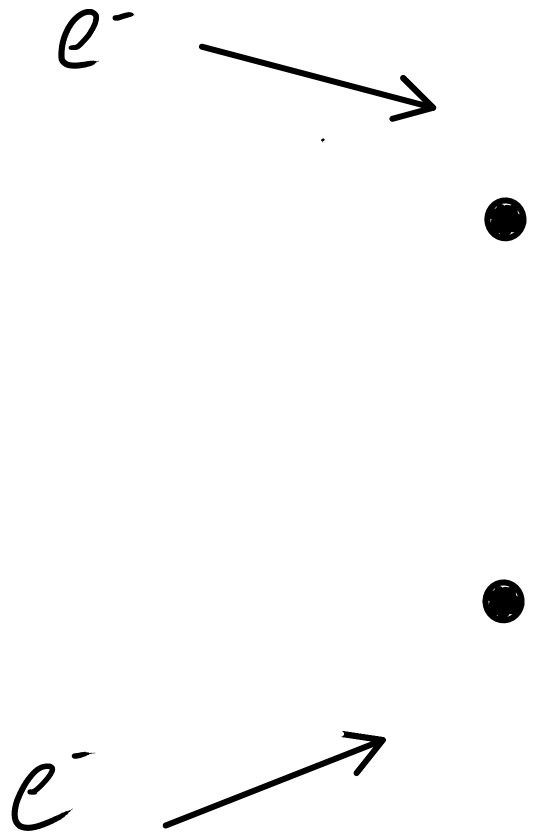

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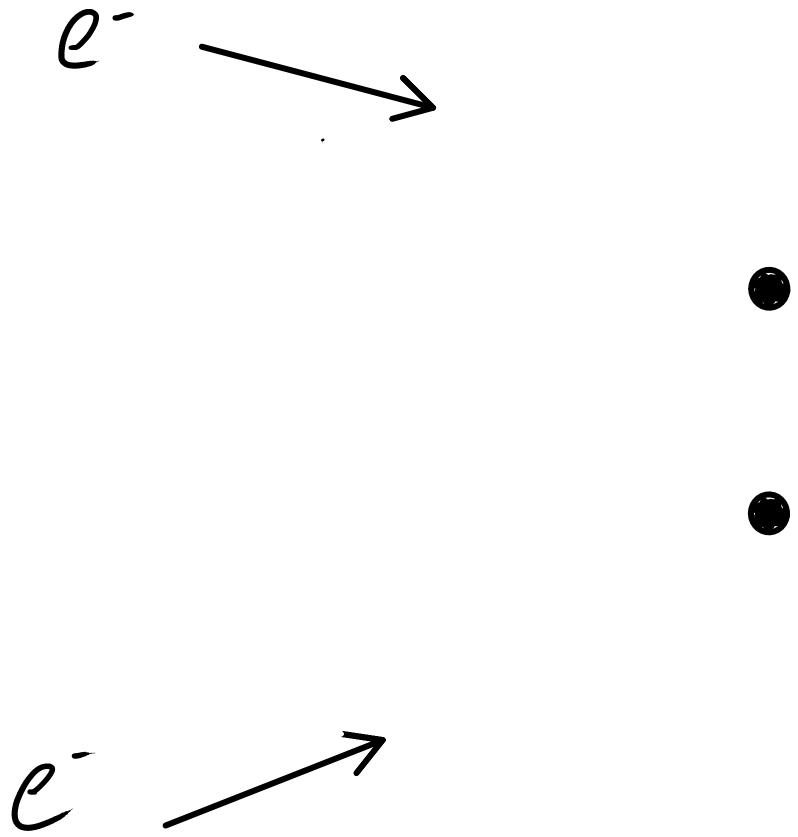

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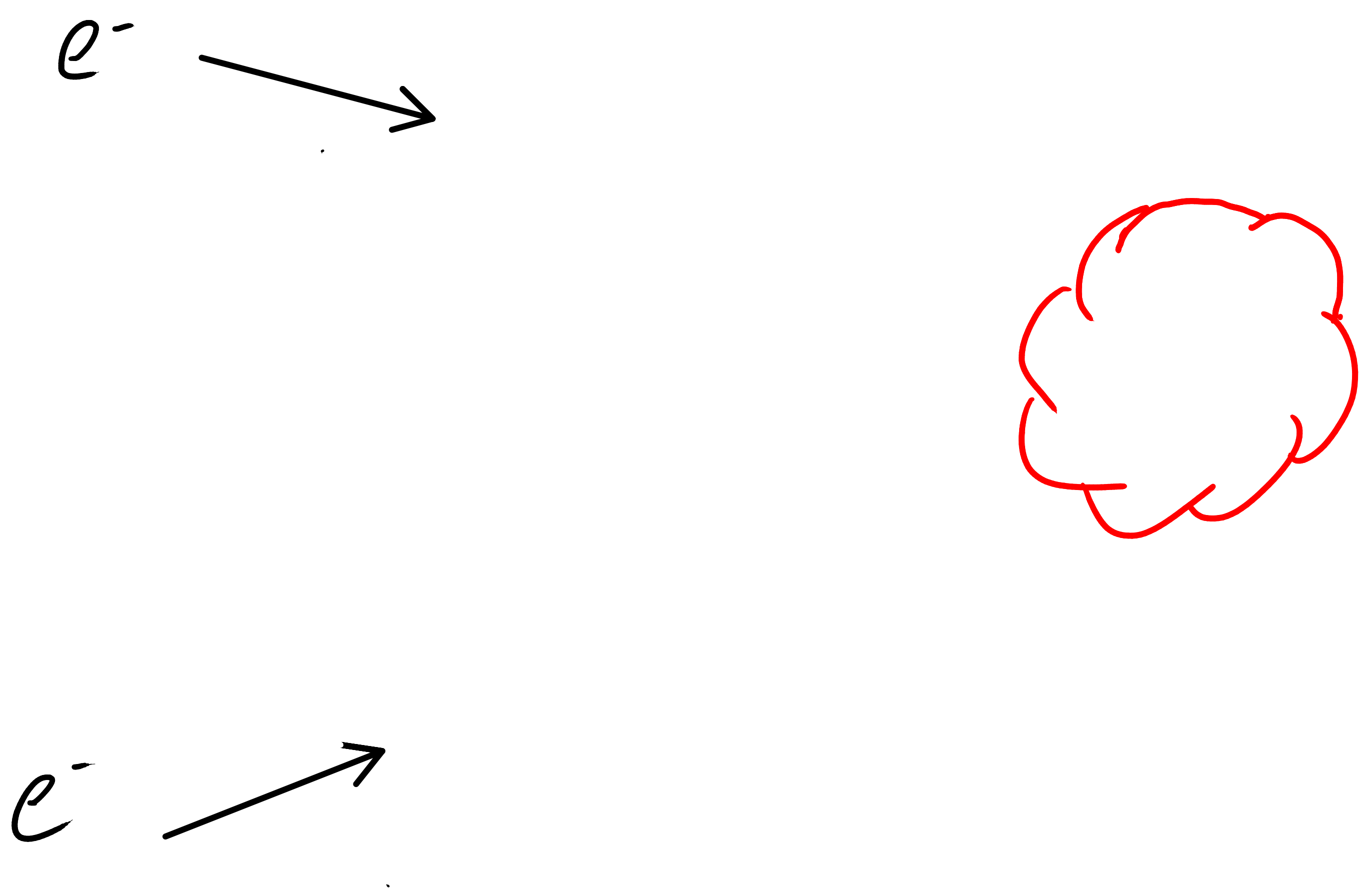

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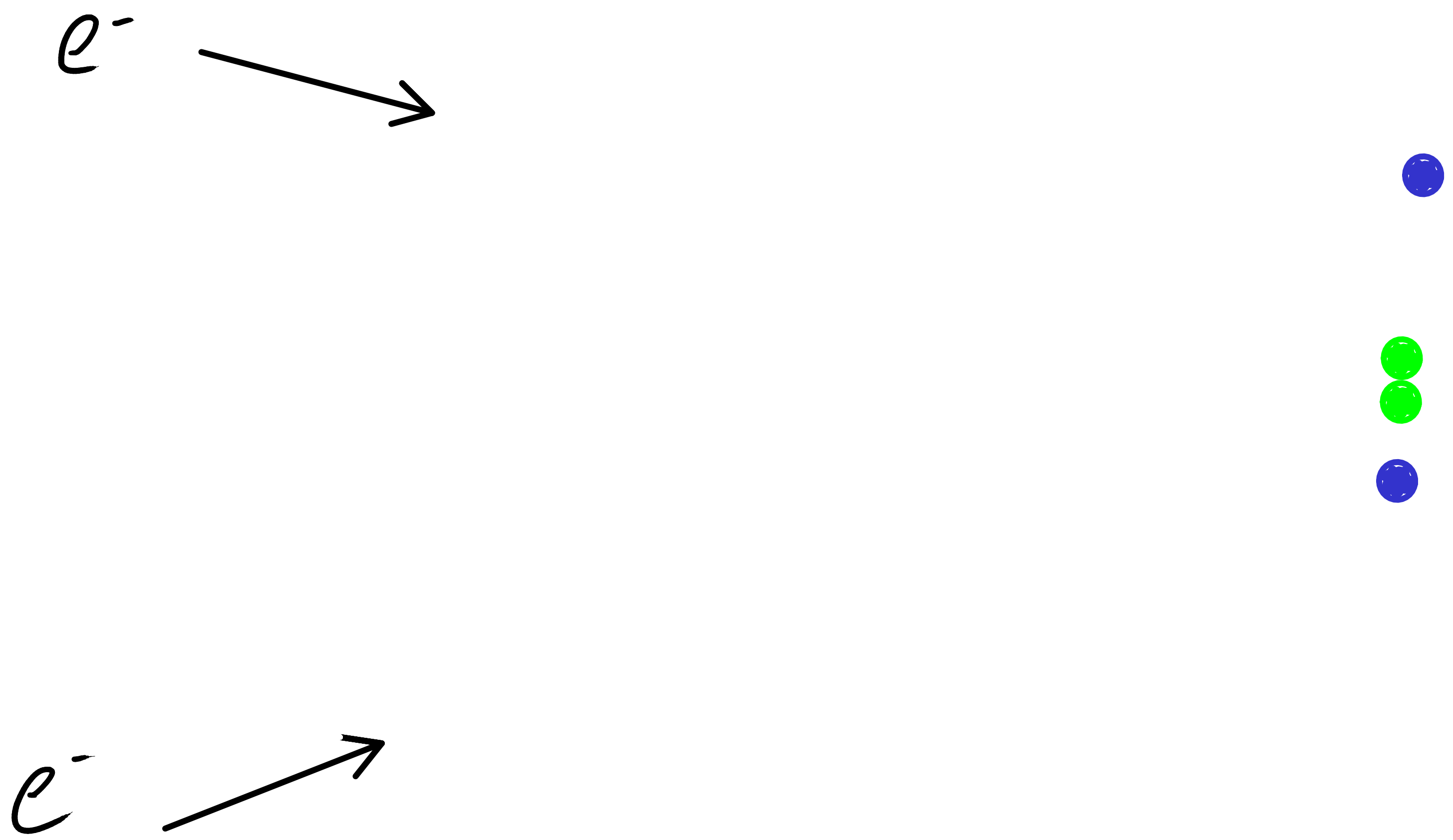

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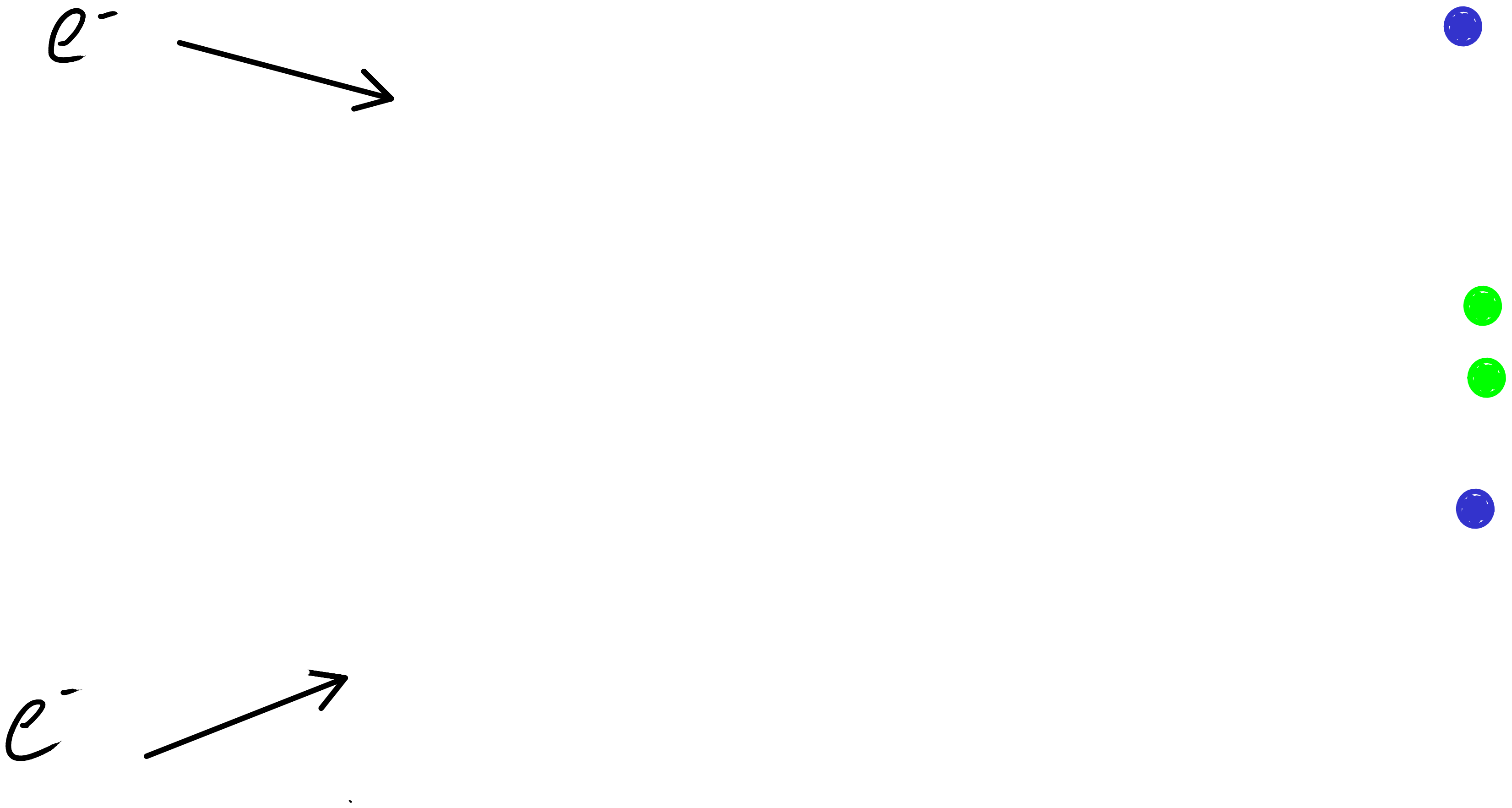

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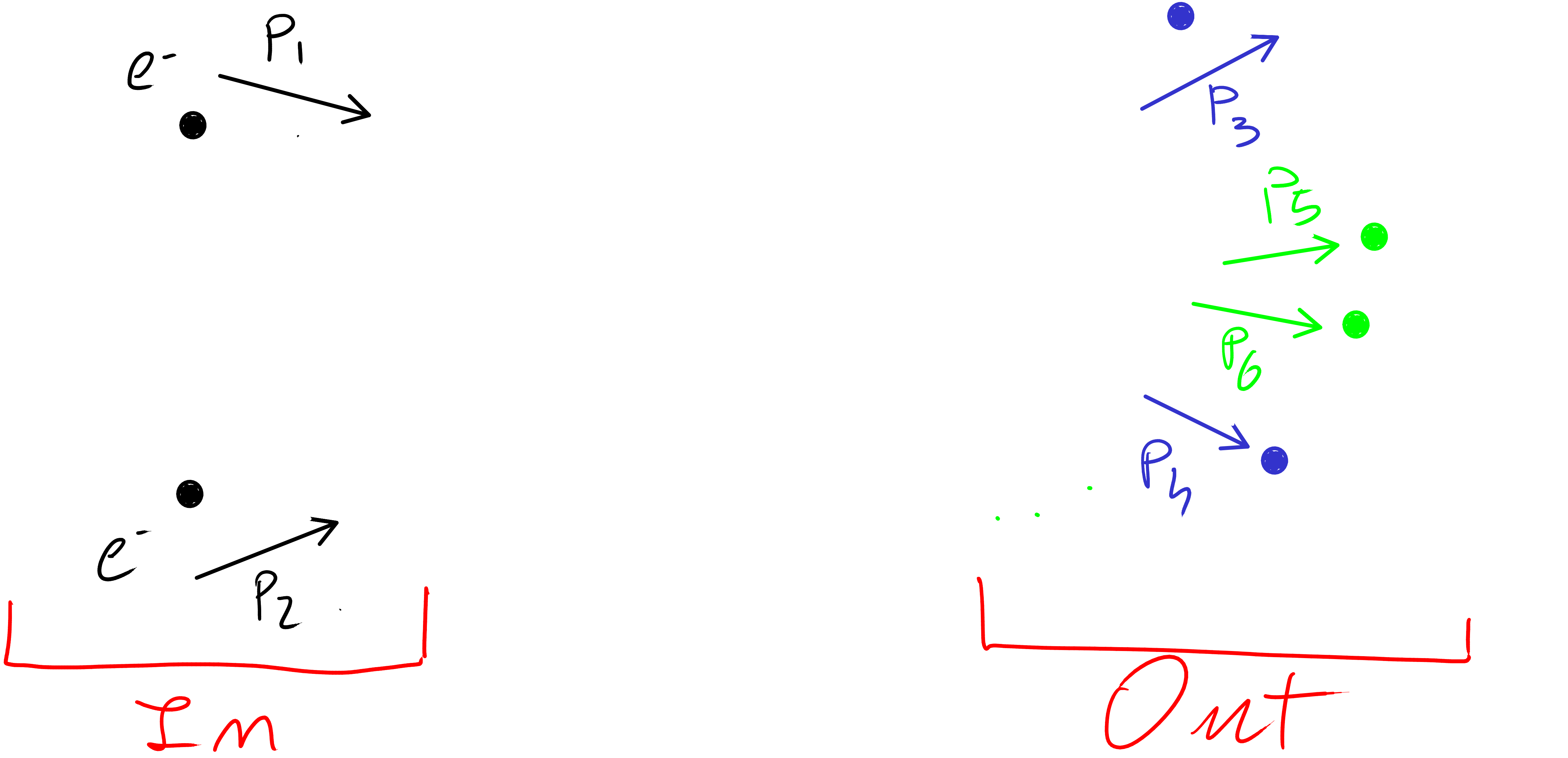

Quantum Physics

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$$P(In, Out) = A(In, Out)^2$$

$$A(In, Out) = \sum_{\gamma \text{ path}} e^{iS(\gamma)}$$

QFT point of view

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QFT point of view

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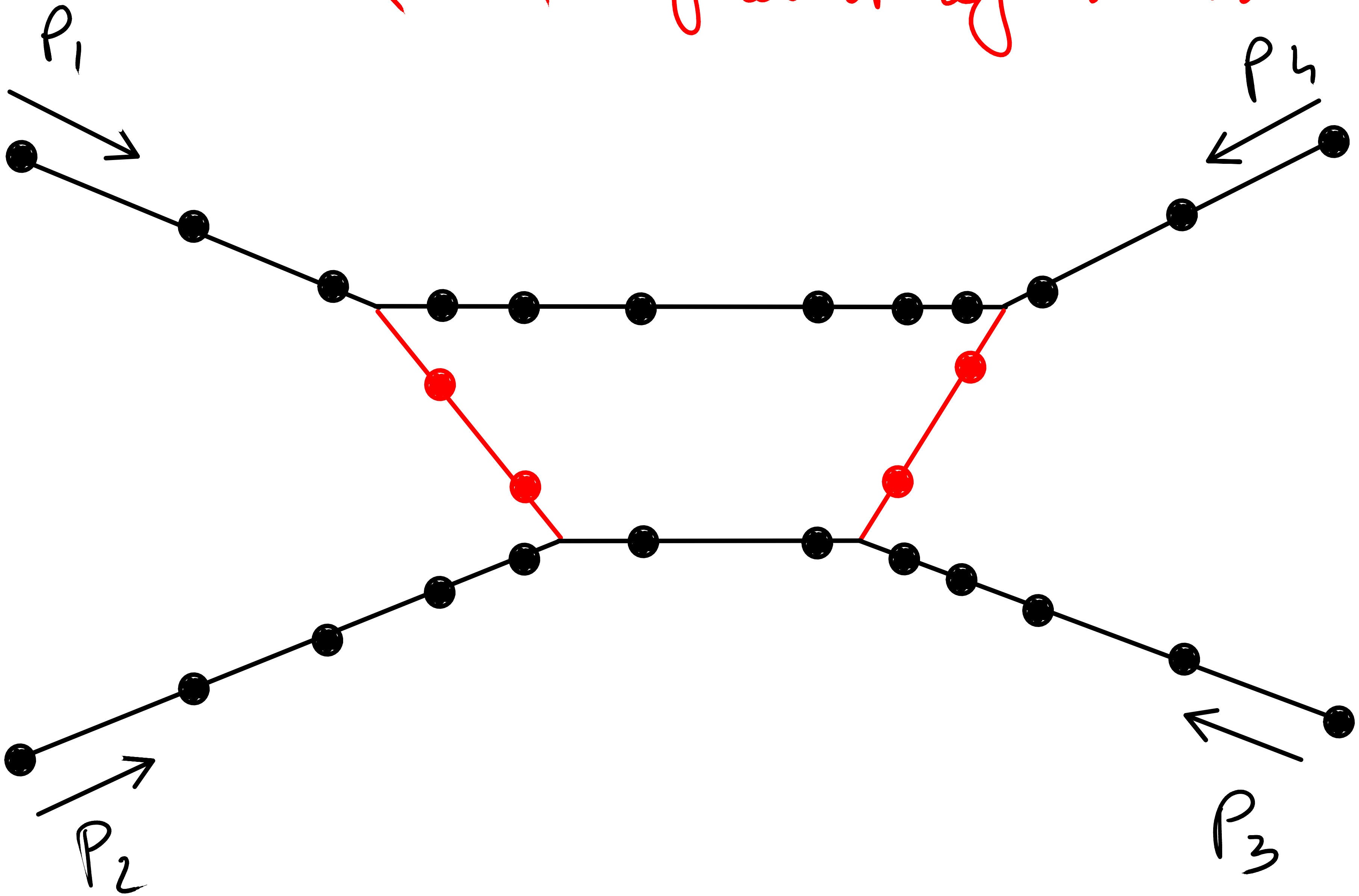
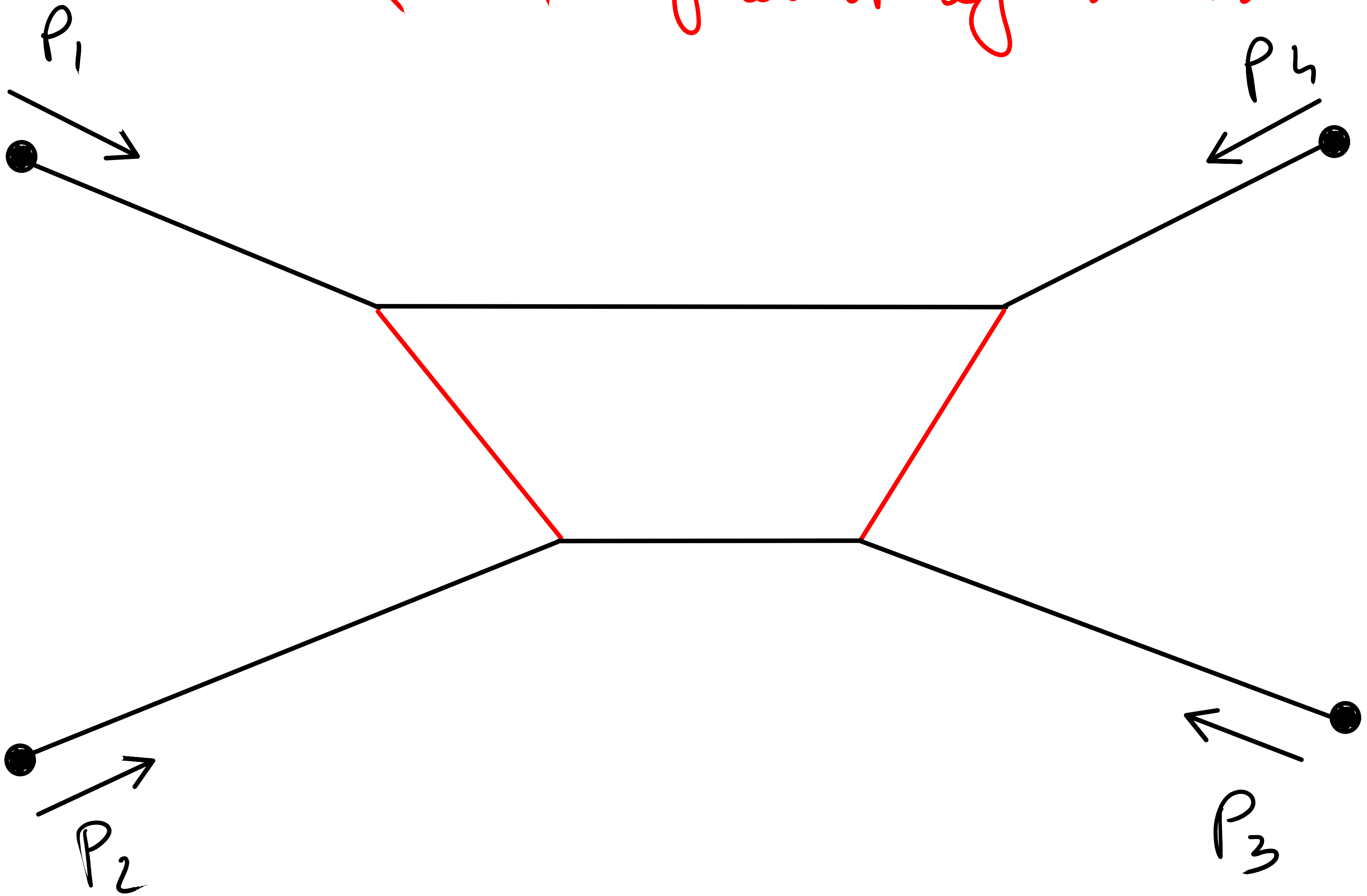
QFT point of view



QFT point of view

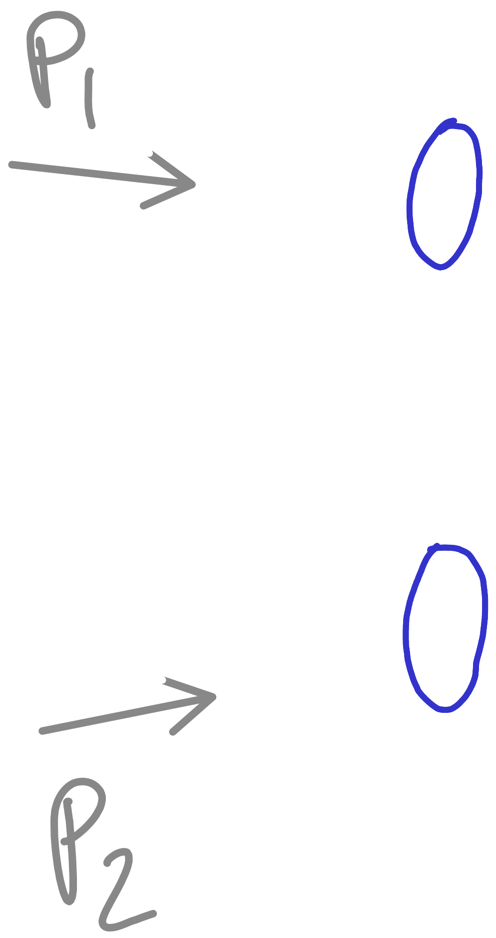


The path is a graph with legs marked with p_i's

String theory point of view.

In

String theory point of view.



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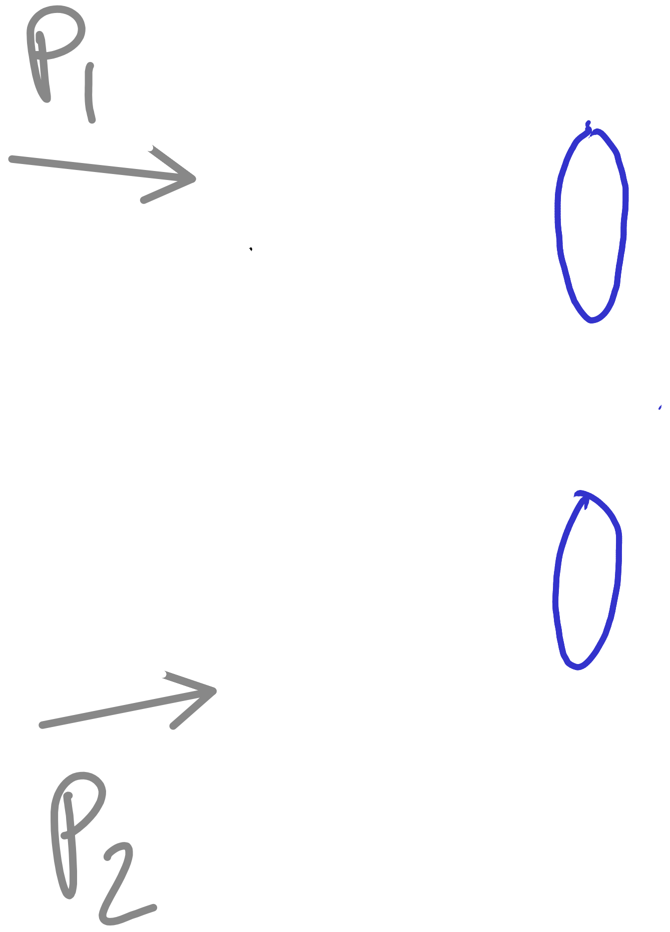

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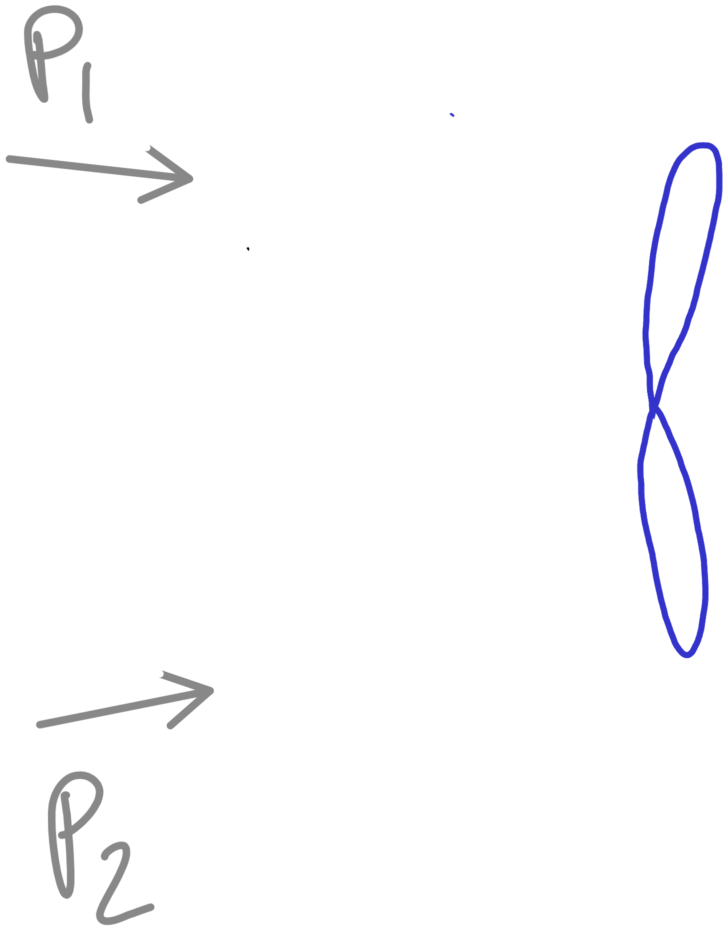

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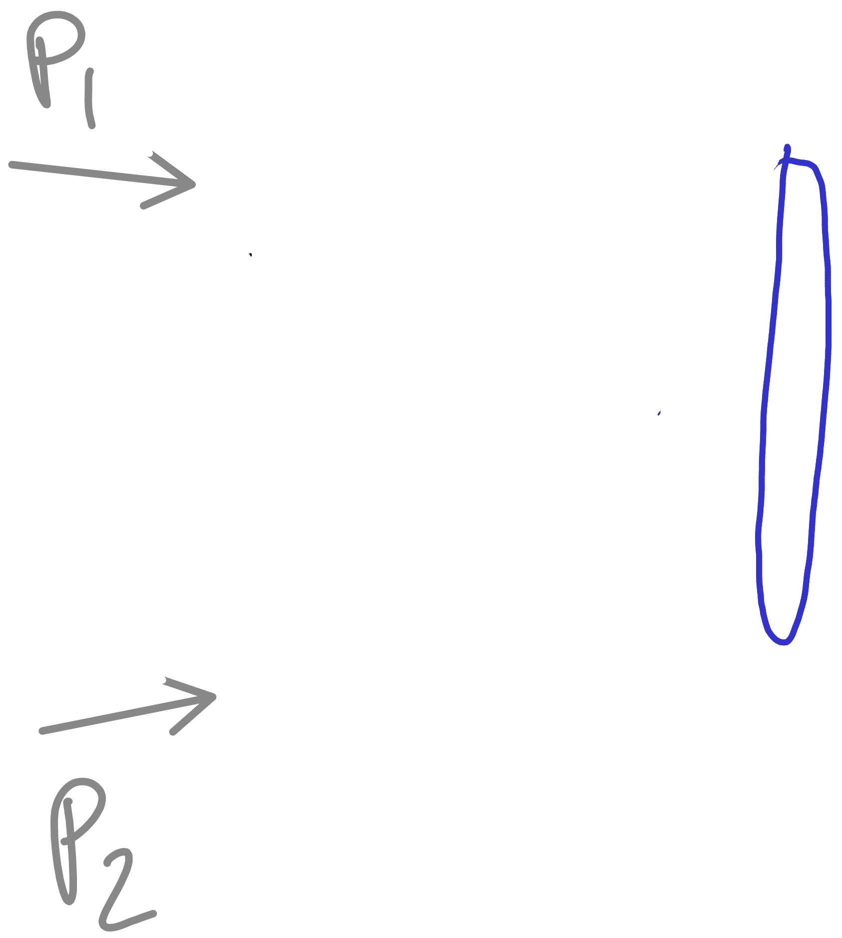

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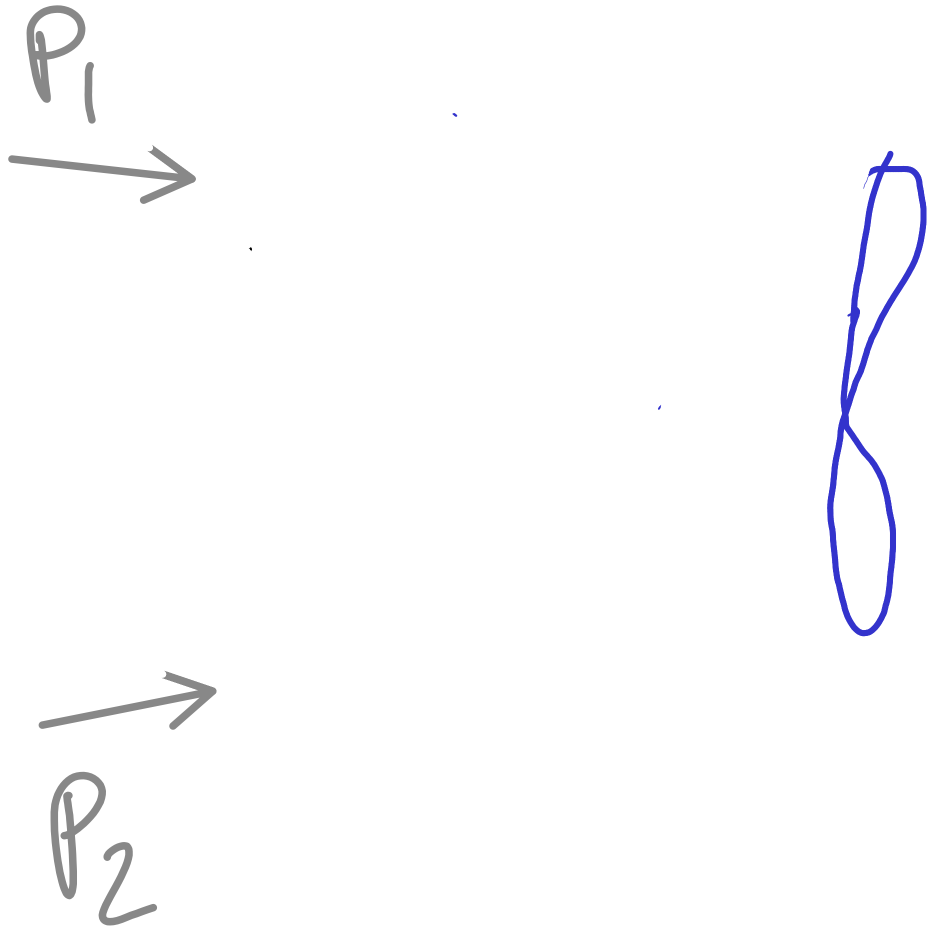

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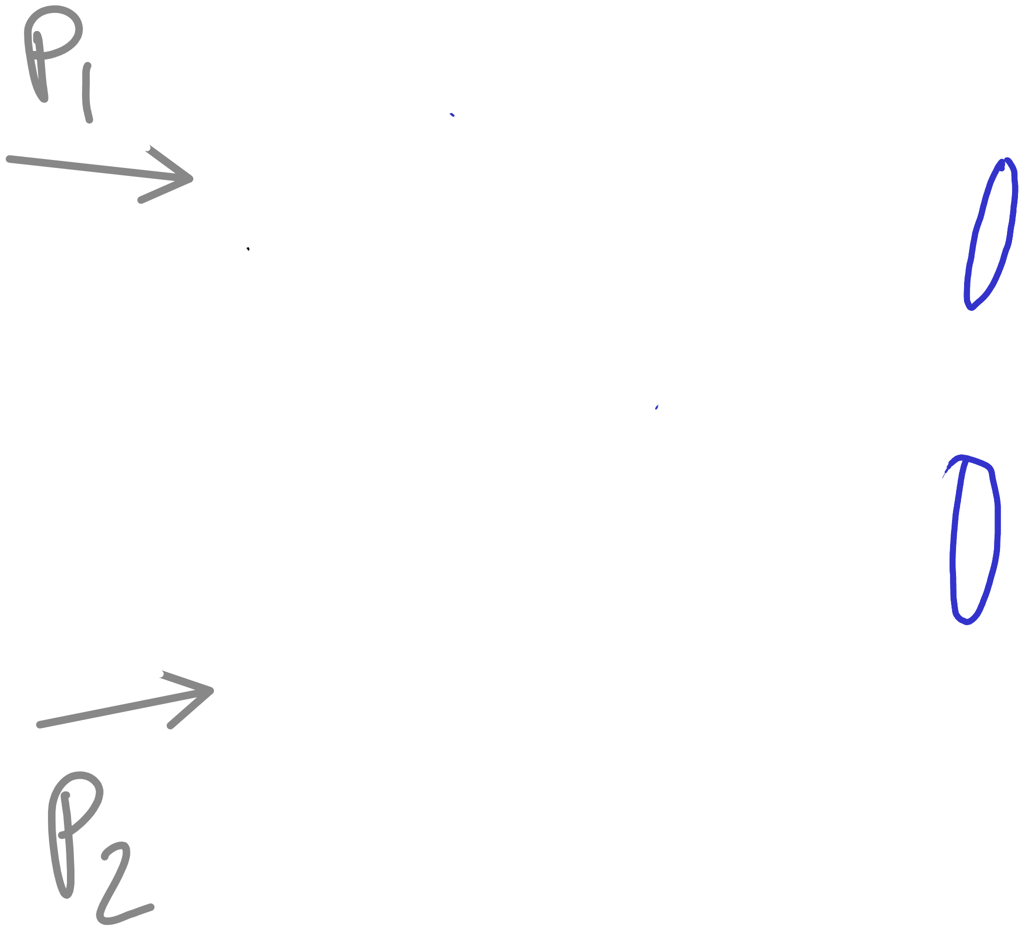

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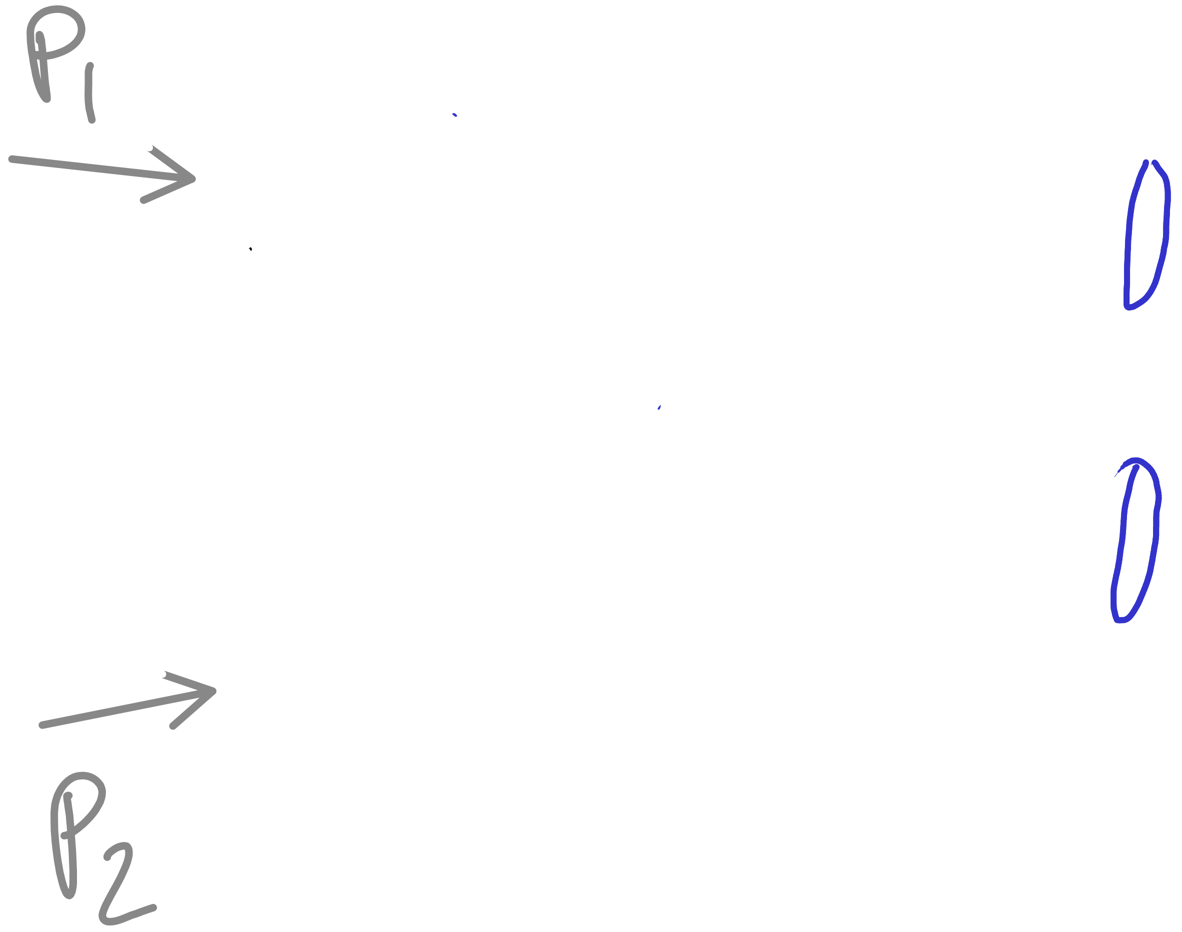

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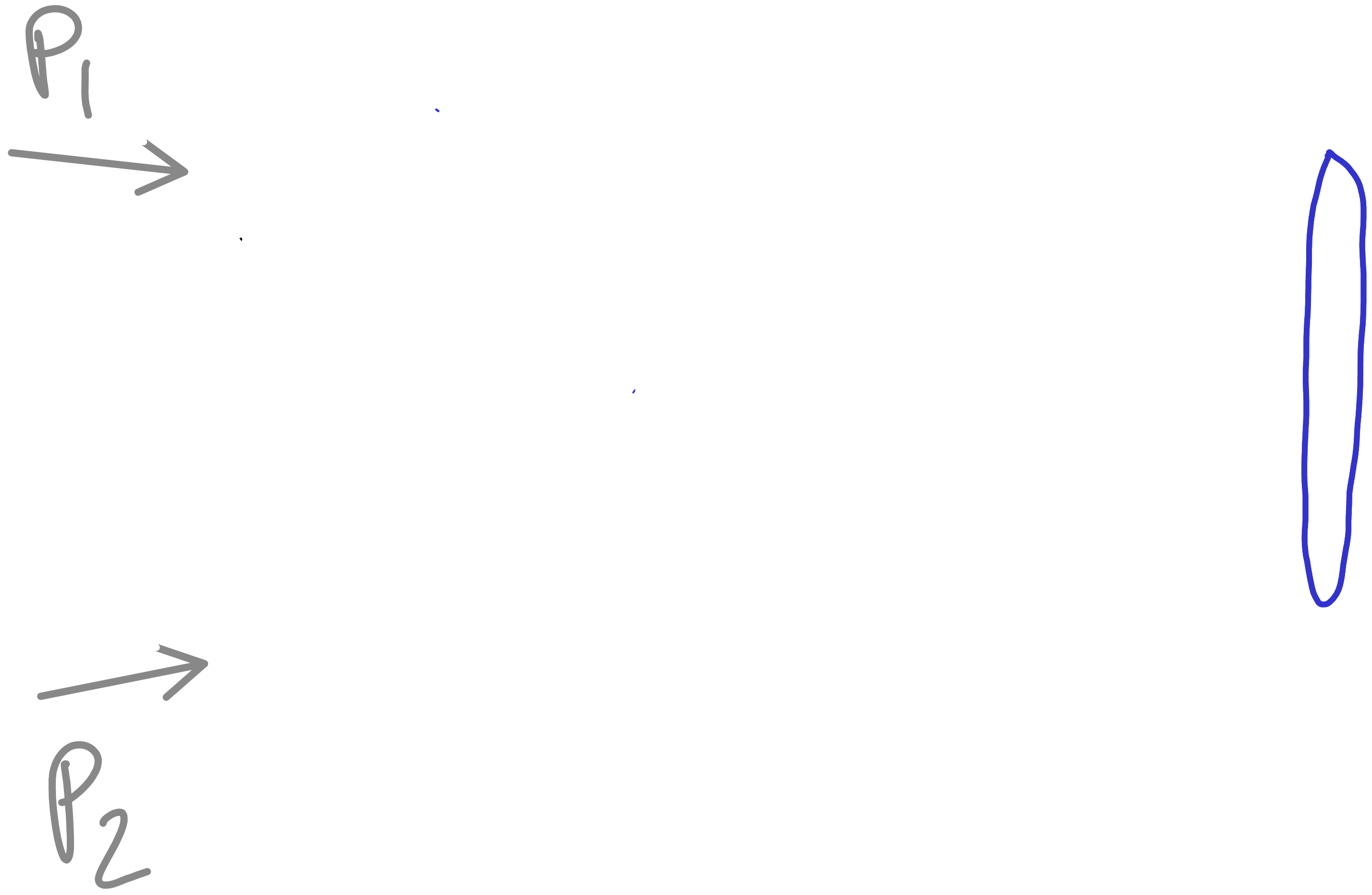

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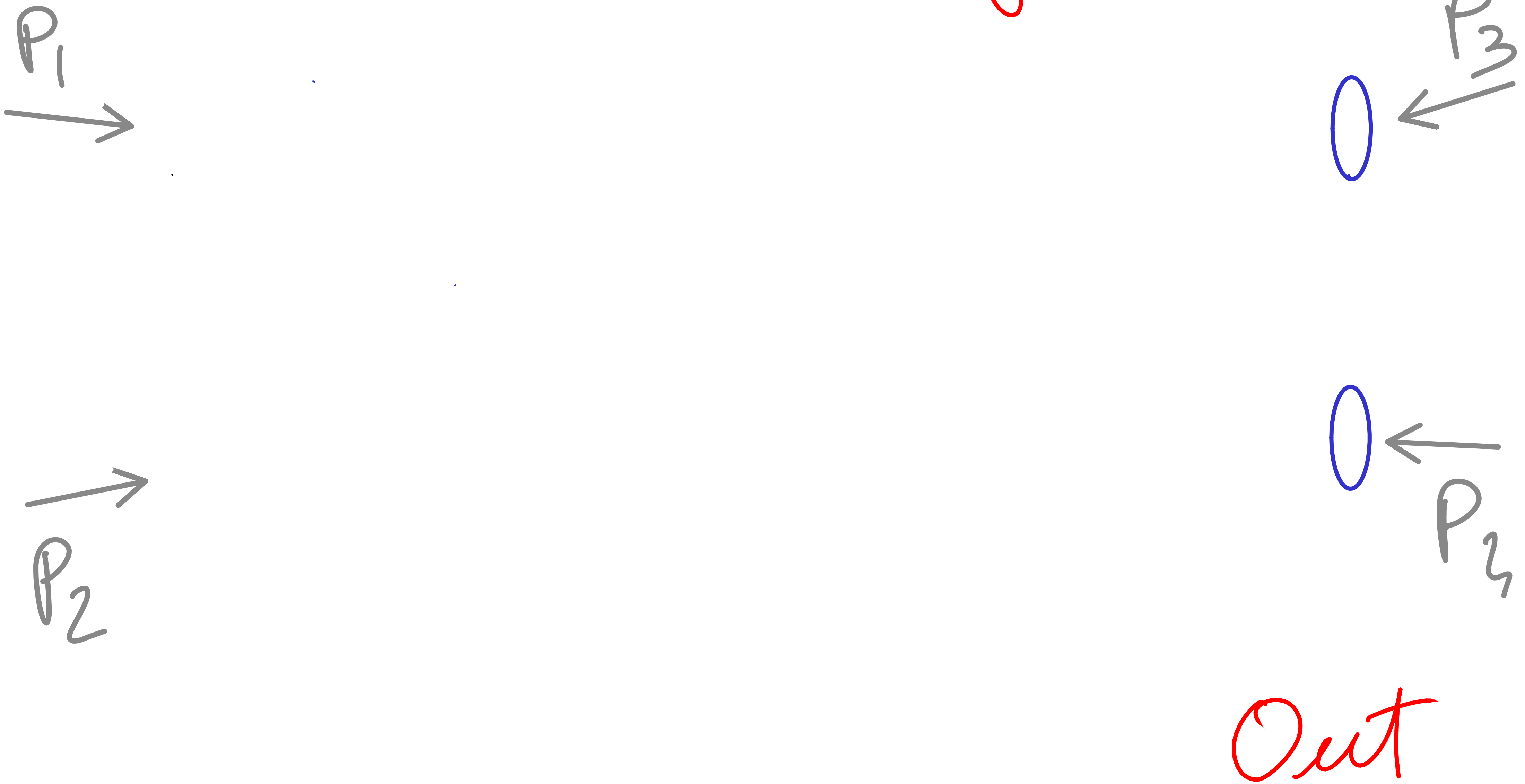

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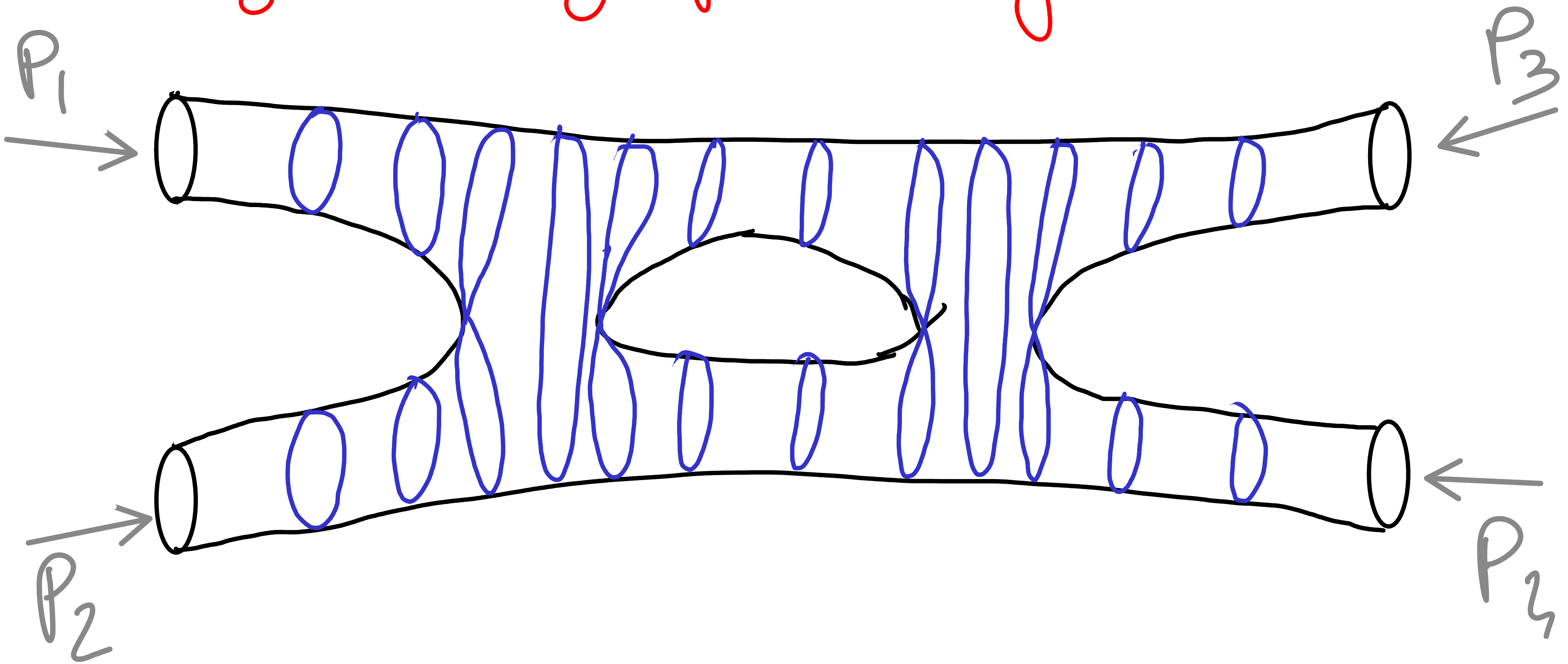

A path is a Riemann surface with punctures
Each marked with a P_i

Notes:

* P_i are "relativistic momentum" $= (E, P)$

$P_i \in \mathbb{R}^D$ with a Minkowski metric D dimension

* By changing the sign we can mix In and Out in a single set $\underline{P}$

* Momentum conservation

$$\sum_{P_i \in \underline{P}} P_i = 0$$

Notes:

* We will consider a simple theory with only one kind of particles. Moreover the particles have no mass.

* On Shell condition

$$m=0 \quad \Rightarrow \quad p_i^2 = \langle p_i, p_i \rangle = 0$$

Minkowski metric.

Notes:

* The string theory has a parameter α' "length" of the string.

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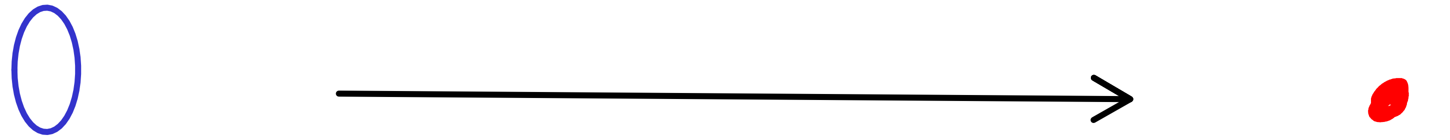
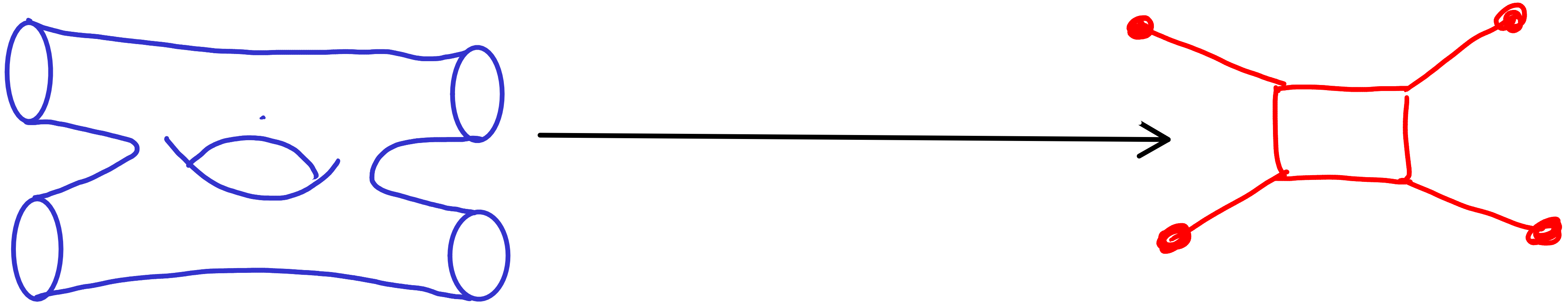
$$\alpha' \longrightarrow 0$$





Notes:

* We classify the possible paths by complexity
  * For graphs this means the number of loops.
  * For Riemann surfaces this means the genus.

QFT Amplitudes

Marked graph: A graph Γ with

$L(\Gamma)$: legs (vertices of order 1)

$V(\Gamma)$ internal vertices (vertices of order ≥ 3)

$E(\Gamma)$ internal edges (edges between int. ver.)

* Each leg i has a label $p_i \in \mathbb{R}^D$

* Each edge e has a length $x_e \in \mathbb{R}$

QFT Amplitudes

$\underline{P} = (p_1, \dots, p_n)$ Momenta (In and Out)

$A^{QFT}(\underline{P}) = \dots =$

$$\sum_{g} \lambda^{g} \sum_{\substack{\Gamma \\ h_1(\Gamma)=g \\ \# L(\Gamma)=n}} N(\Gamma) \int_{\mathbb{R}_+^{E(\Gamma)}} e^{i \frac{\varphi_\Gamma(\underline{x_e}, \underline{P})}{\psi_\Gamma(x_e)}} \frac{\prod dx_e}{(\psi_\Gamma(x_e))^{D/2}}$$

λ is the coupling constant. Depends on the theory and should be small.

$h_1(\Gamma)$ Number of loops of Γ

$N(\Gamma)$ Normalization factor. Depends on the theory and takes into account the automorphisms of Γ

ψ_Γ First Symanzik polynomial

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First Symanzic polynomial

$$\Psi_\Gamma(x_e) = \sum_{\substack{T \\ \text{spaning} \\ \text{tree}}} \prod_{e \notin T} x_e$$

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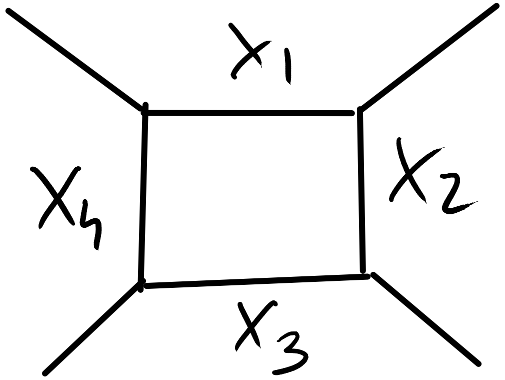
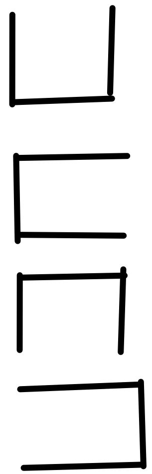
Example:



Γ



Trees	Term
	x_1
	x_2
	x_3
	x_4

$$\Psi_\Gamma(x_e) = x_1 + x_2 + x_3 + x_4$$

String amplitudes

$$A^{\text{string}}(\underline{P}, \alpha') = \cdots =$$

$$\sum_g \lambda^g N(n, g, \alpha') \int_{\mathcal{M}_{g,n}} e^{i\alpha' \sum_{1 \le k, l \le n} \langle P_k, P_l \rangle g_c(x_k, x_l)} d\mu(c, x_k)$$

String amplitudes

$$\mathcal{A}^{\text{string}}(\underline{P}, \alpha') = \cdots = \sum_g \lambda^g N(n, g, \alpha') \int_{\mathcal{M}_{g,n}} e^{i\alpha' \sum_{1 \le k,l \le n} \langle P_k, P_l \rangle g_c(x_k, x_l)} \, d\mu(c, \underline{x})$$

Koba Nielsen factor

A measure.

* $(C, \underline{x})$ are coordinates of $\mathcal{M}_{g,n}$:

C curve

$\underline{x} = (x_1, \ldots, x_n)$ are the marked points

* g_C Green function on C.

* Conditions $\sum p_i = 0$ $\langle p_i \, p_i \rangle = 0$

imply that $\sum_{1 \le k, \ell \le n} \langle p_k, p_\ell \rangle \, g_C(x_k, x_\ell)$ is

well defined and finite.

Observation

$\sum_{1\le k,\ell\le m} \langle p_k, p_\ell \rangle g_c(x_k, x_\ell)$ can be seen as a height pairing of vector valued divisors

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Question

How can we understand

$$\lim_{\alpha' \to 0} A^{\text{string}}(\alpha', \underline{p}) = A^{QFT}(\underline{p})$$

Moduli space of stable marked graphs

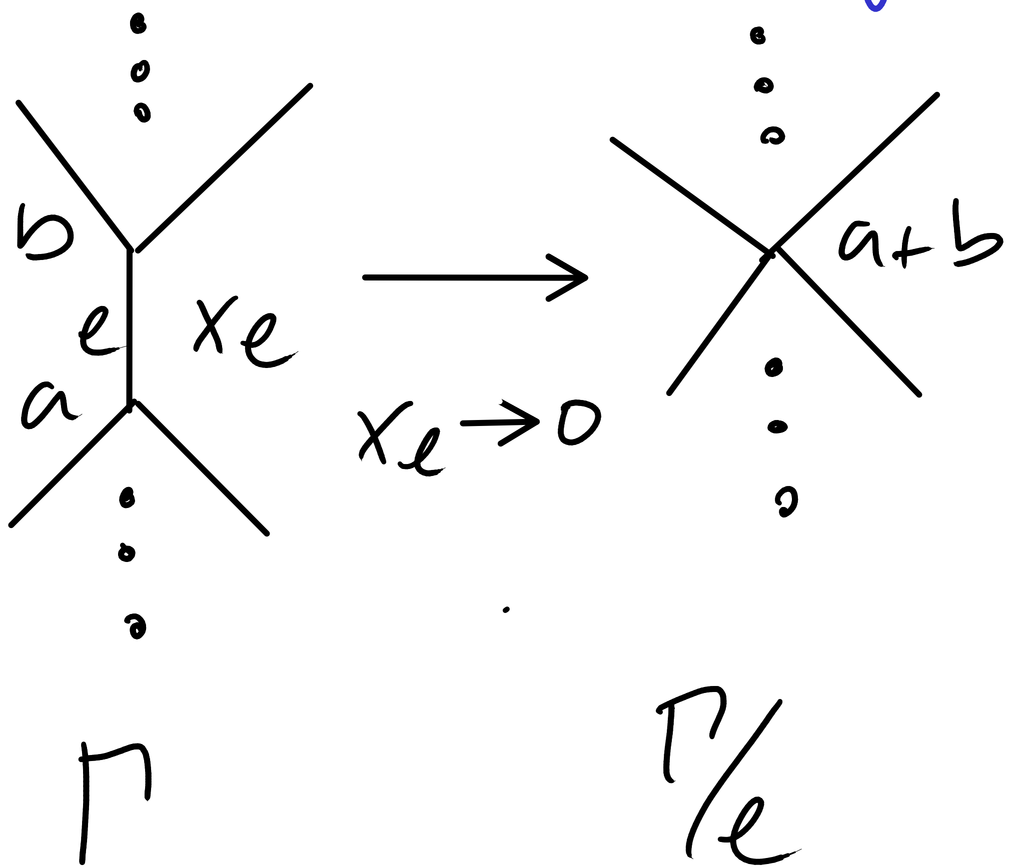
Stable marked graphs aka Tropical curves

* A graph with legs, vertices and edges
* Each leg has a label
* Each vertex v has a local genus $g_v \geq 0$
* Each edge e has a length x_e
* is connected
* each vertex v with $g_v = 0$ has index ≥ 3

The genus is defined as $g(\Gamma) = h_1(\Gamma) + \sum_v g_v$

Moduli space of stable marked graphs

Contraction operation:



Moduli space of stable marked graphs

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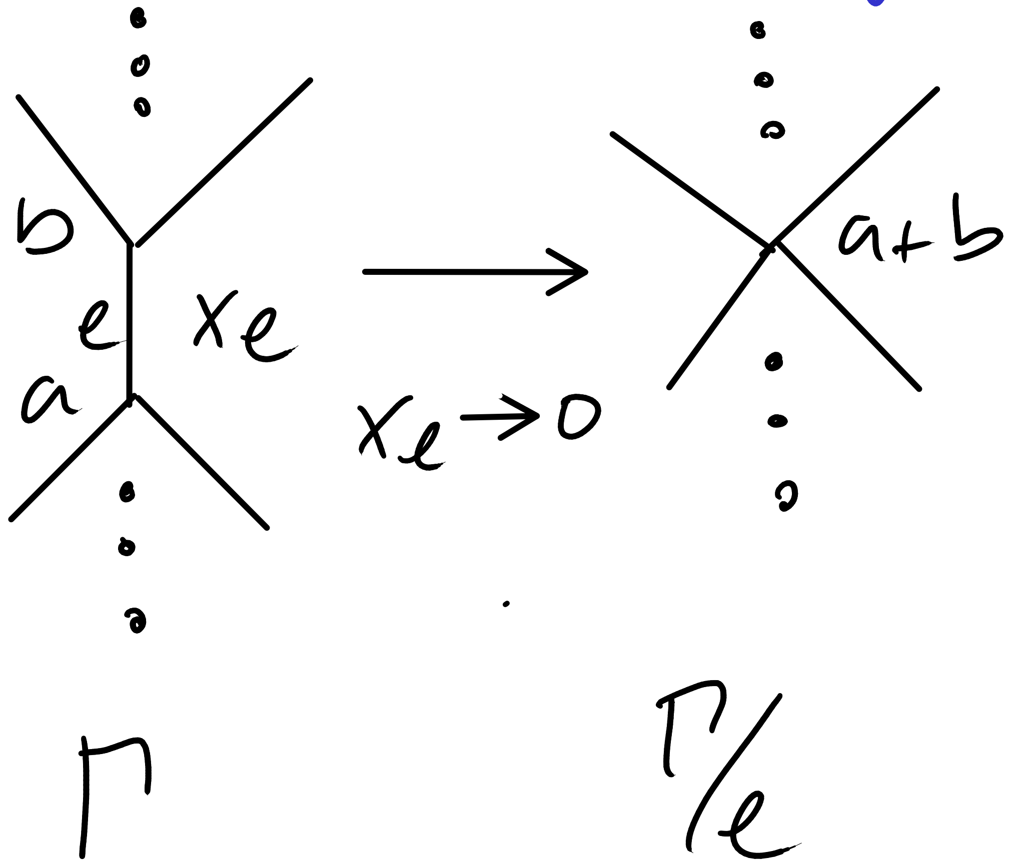
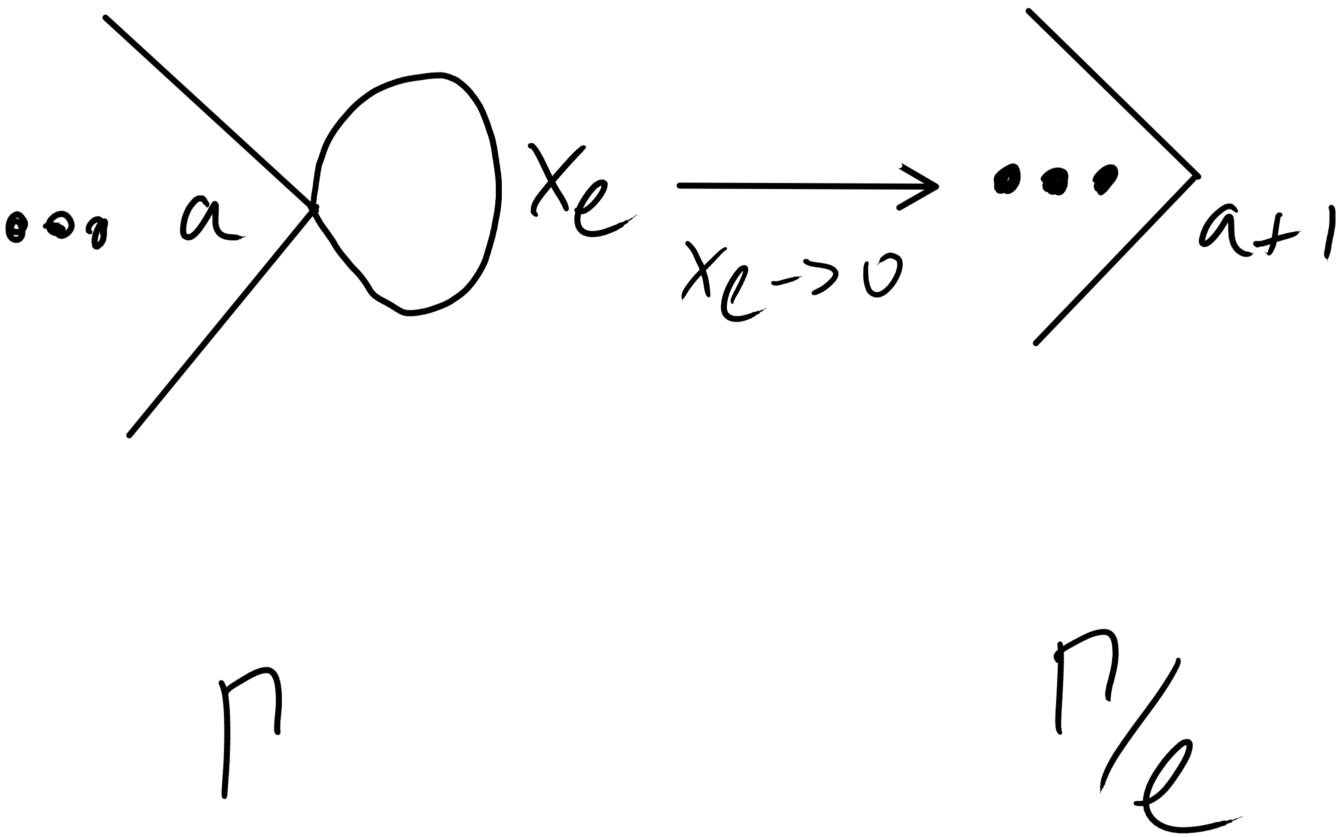




Moduli space of stable marked graphs

$$\mathcal{M}_{g,n}^{trop} = \left(\coprod_{\substack{\Gamma \text{ stable} \\ g(\Gamma)=g \\ \# L(\Gamma)=n}} \mathbb{R}_+^{E(\Gamma)} \Big/ \sim \right) \Big/ G_\Gamma$$

The first relation identifies Γ with $x_e = 0$ to Γ/e

The second relation identifies automorphisms

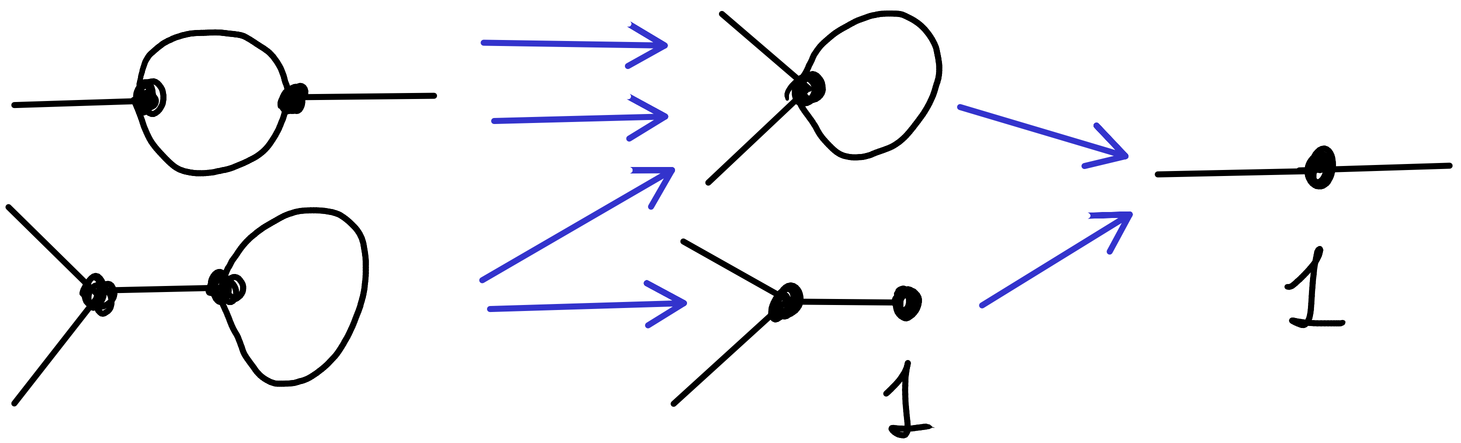
Moduli space of stable marked graphs

The QFT amplitude can be written as

$$A^{QFT} = \sum_g \lambda^g \int_{\mathcal{M}_{g n}^{trop}} e^{i \frac{\varphi(\underline{x}, \underline{p})}{\psi(x)}} d\mu$$

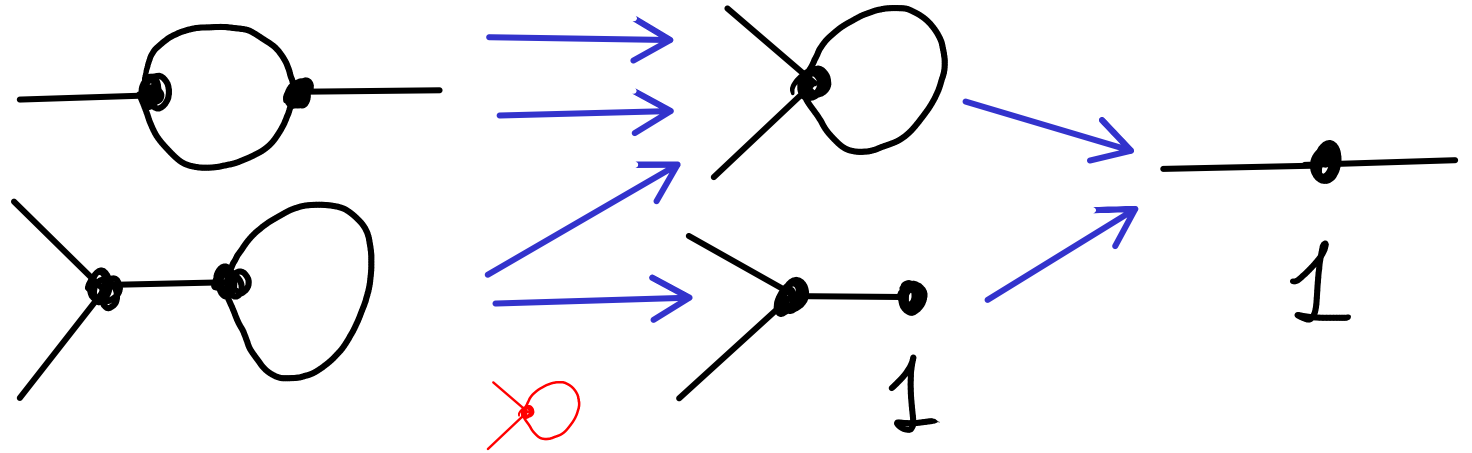
Example M^{trop}_{12}

Graphs

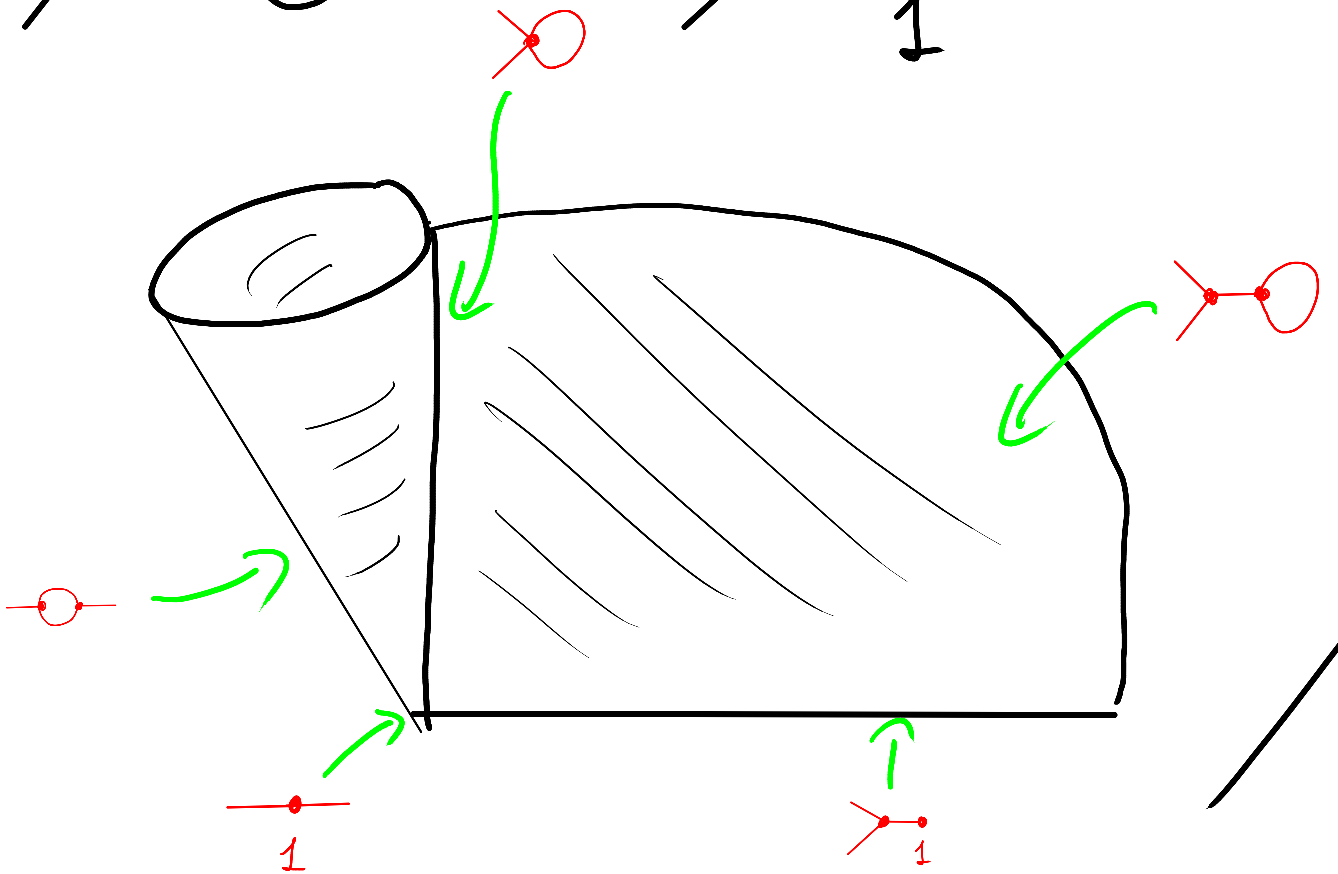


Example $\mathcal{M}^{trop}_{1,2}$

Graphs



Moduli



$/\ \widetilde{G}(\Gamma)$

Moduli of stable curves

$\mathcal{M}_{g,n}$ can be compactified to the moduli space of stable curves.

* connected and at worst node singularities
* n smooth marked points
* each irred. component has at least

$g \geq 2$

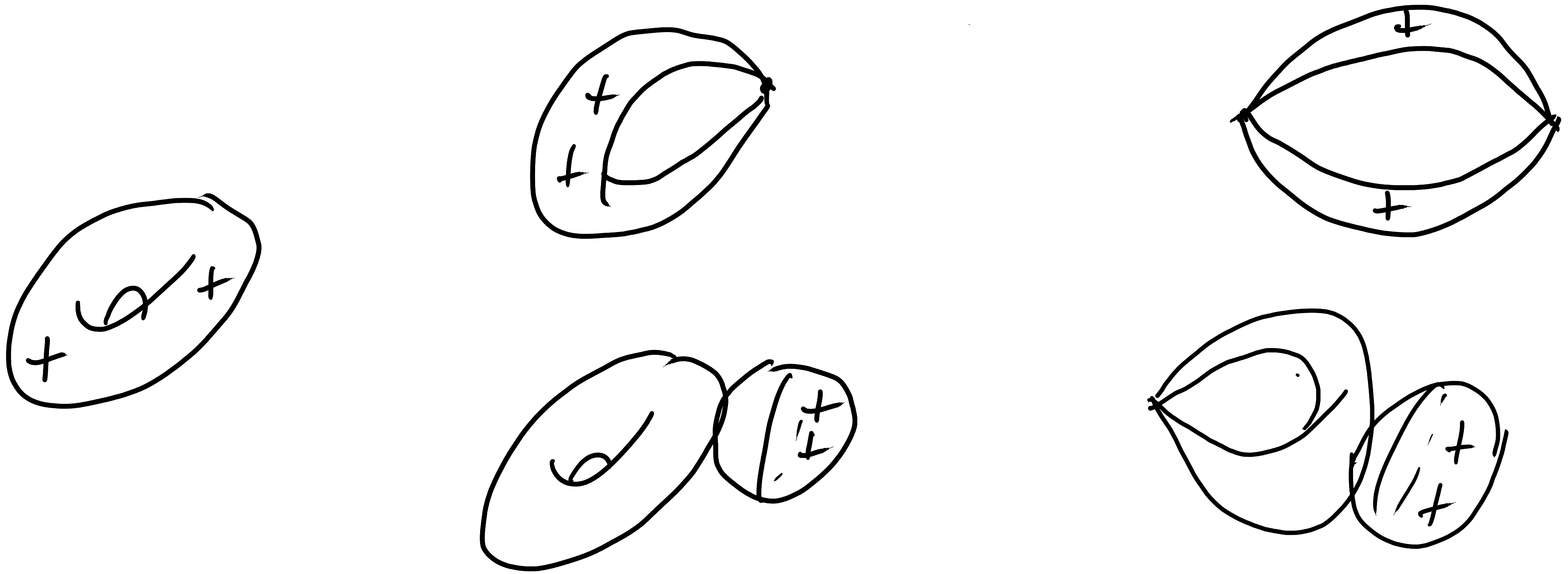
or $g = 1$ and ≥ 1 exceptional points

$g = 0$ and ≥ 3 exceptional points

exceptional point: marked or singular.

Example $\overline{M}_{12}$

curves



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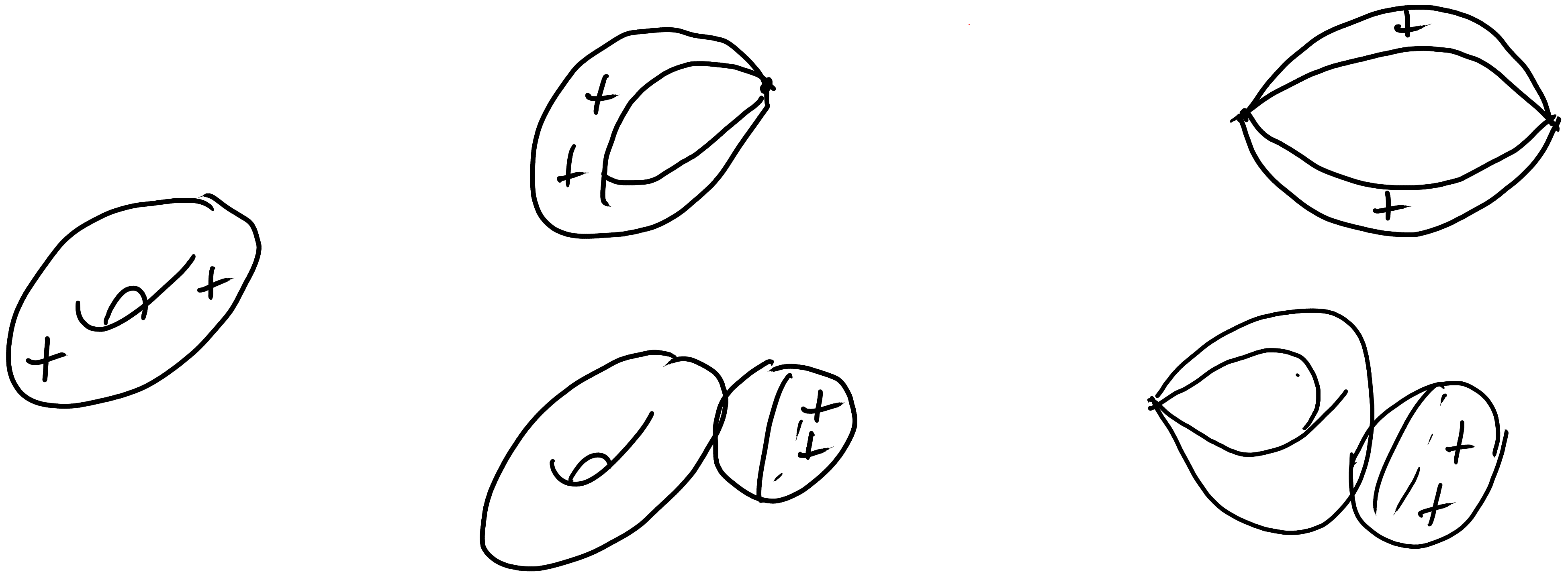
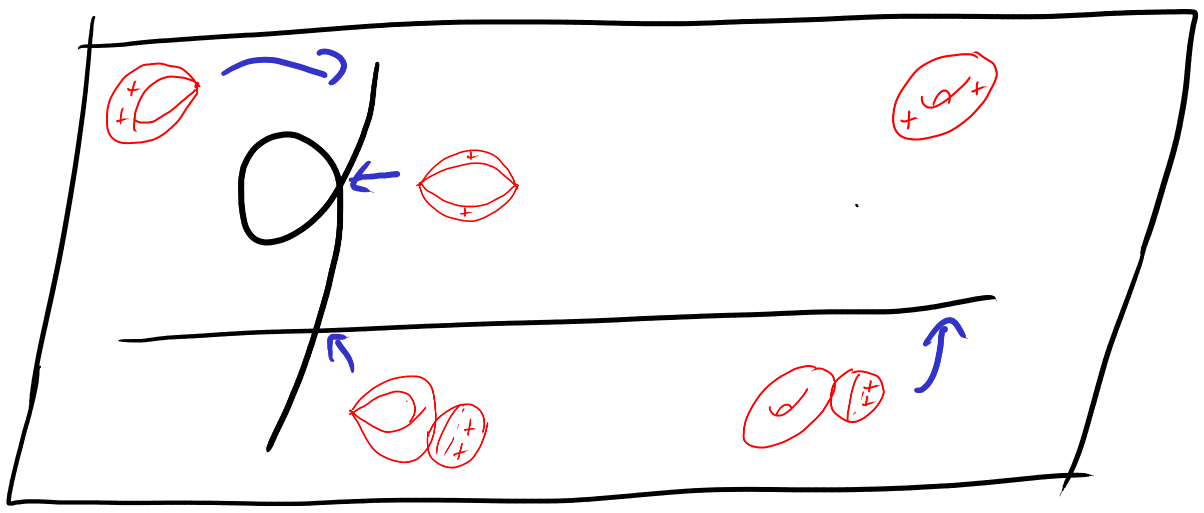


Moduli



A Correspondence

Stable marked curves $\qquad$ Stable marked graphs

irreducible component of genus g $\longrightarrow$ vertex of genus g

singular point $\longrightarrow$ edge.

marked point $\longrightarrow$ leg

This correspondence reverses dimension.

Tropicalization

$X \subseteq \bar{X}$ smooth compactification

$D = \bar{X} \setminus X$ a simple normal crossing divisor

$D = D_1 \cup \cdots \cup D_r$ D_i smooth divisors

$J \subseteq \{1, \ldots, r\}$ $D_J = \bigcap_{i \in J} D_i$ Assume is irreducible.

$C_J = (\mathbb{R}_+)^{\# J}$

$$\mathrm{Trop}(X, \bar{X}) = \left(\coprod_{\substack{J \subseteq \{1, \ldots, m\} \\ D_J \neq \phi}} C_J \right) \Big/ \sim \quad \text{Face relations}$$

Tropicalization

Generalizes to toroidal embeddings of D-M-Stacks

Theorem (Abramovich-Caporaso-Payne)

$$\mathrm{Trop}\left(\overline{\mathcal{M}}_{g,n}, \mathcal{M}_{g,n}\right) = \mathcal{M}_{g,n}^{trop}.$$

The hybrid Topology

* Berkovich - Jonsson
* Ideas of Bergmann ~71, Morgan - Shalen 84

Idea: X^{trop} is a degeneration of X.

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Idea: X^{trop} is a *degeneration* of X.

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$D = \bar{X} \setminus X$ a simple normal crossing divisor

$$\mathcal{X}^{hyb} = X \times (0,1] \amalg X^{trop} \times \{0\}$$

+ a topology.

The hybrid Topology

Can be seen as a exponential deformation to the normal cone

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Can be seen as a exponential deformation to the normal cone

local coordinates $D = \{ z_1 \cdots z_k = 0 \}$

$p \in D_{\{1,\dots,k\}}$ $\quad p = (\underbrace{0,\dots,0}_{k},*,\dots,*) \quad * \in \mathbb{C}$

$D_{\{1,\dots,k\}} \rightsquigarrow C_{\{1,\dots,k\}} \subseteq X^{trop}$

component $\qquad$ cone

The hybrid Topology

Can be seen as a exponential deformation to the normal cone

Path $\gamma(\alpha') = \left((z_1 e^{-r_1/\alpha'}, \dots, z_k e^{-r_k/\alpha'}, *, \dots, *), \alpha' \right) \in \mathcal{X}^{hyb}$

$r_i > 0$ $\qquad$ * Sub exponential

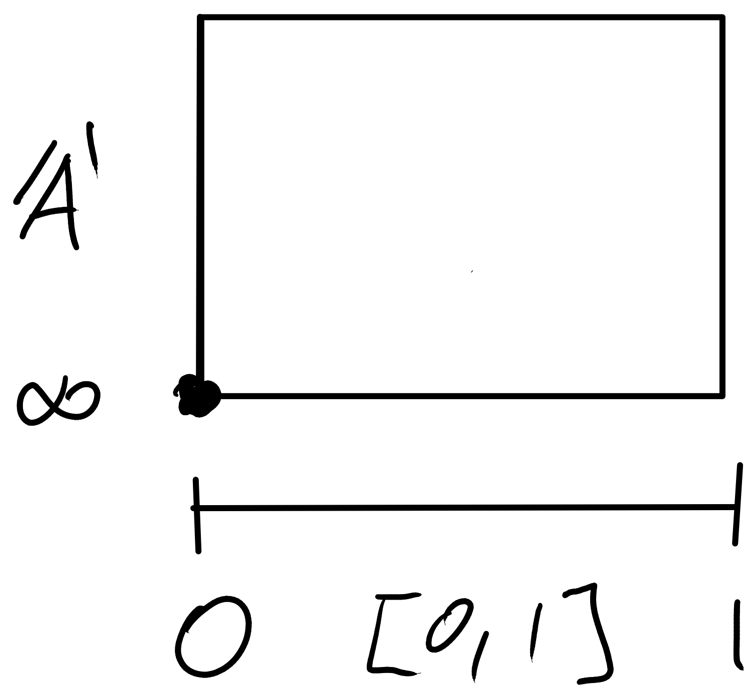
$$\lim_{\alpha' \to 0} \gamma(\alpha') = (r_1, \dots, r_k) \in C_{\{1,\dots,k\}}$$

The hybrid Topology

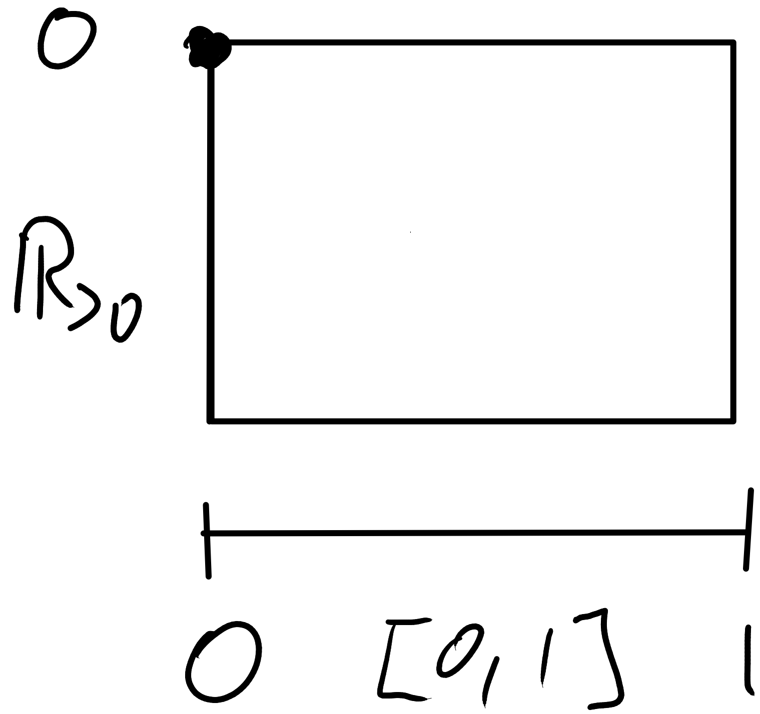
Can be seen as a exponential deformation to the normal cone

Example: $X = \mathbb{A}^1$ $\quad \bar{X} = \mathbb{P}^1$

$\bar{X} \times [0,1]$



$\mathcal{X}^{hyb}$



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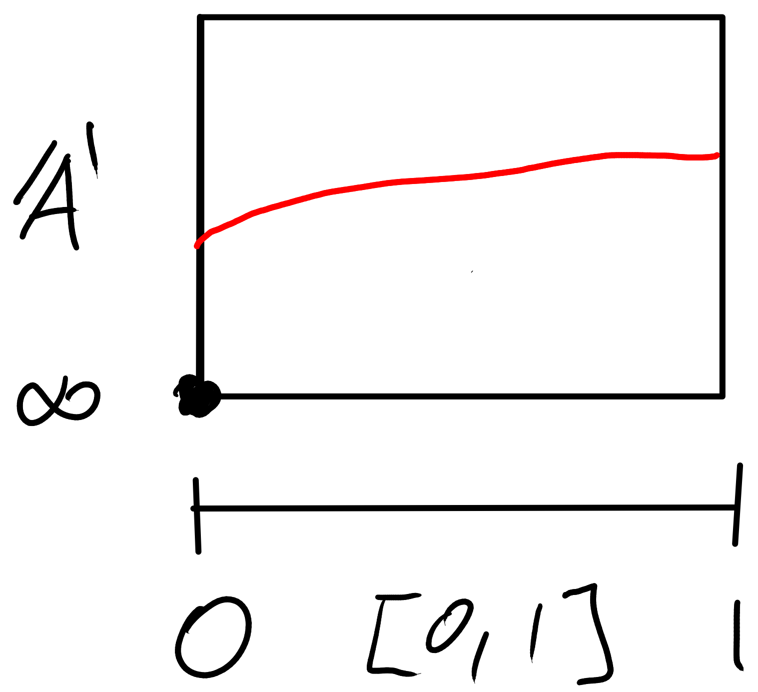
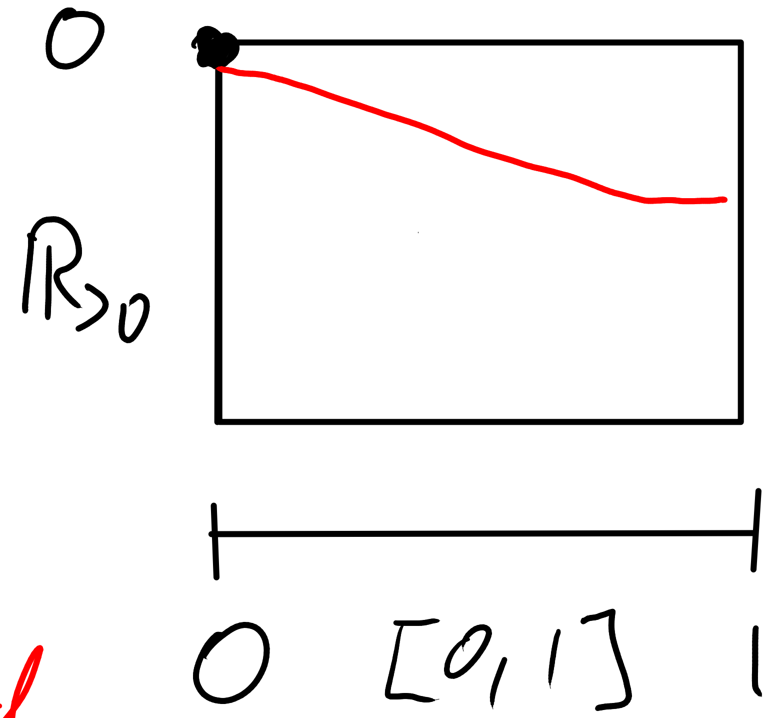


bounded

$\mathcal{X}^{hyb}$



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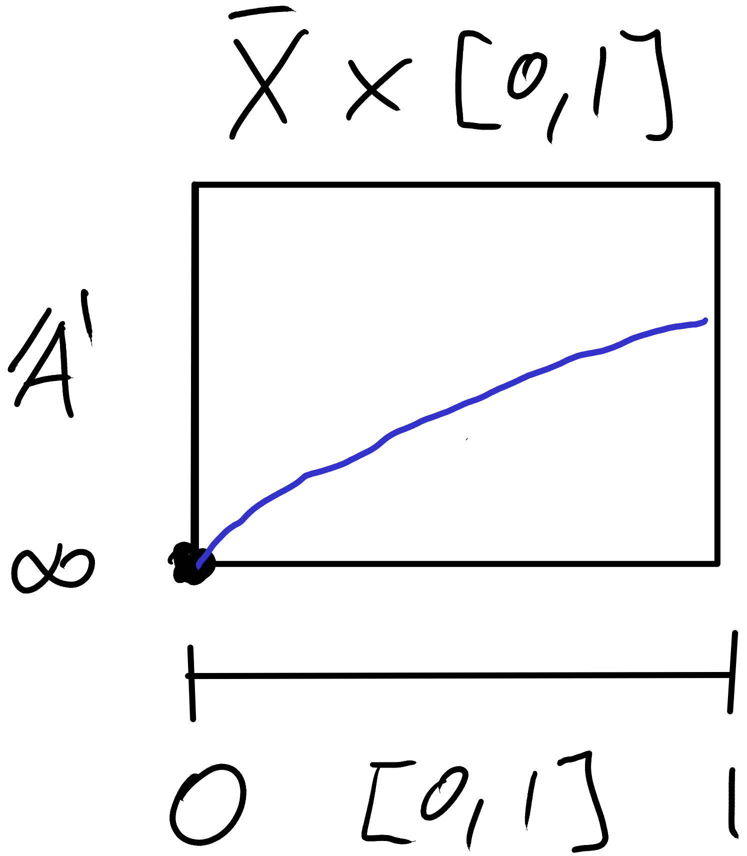
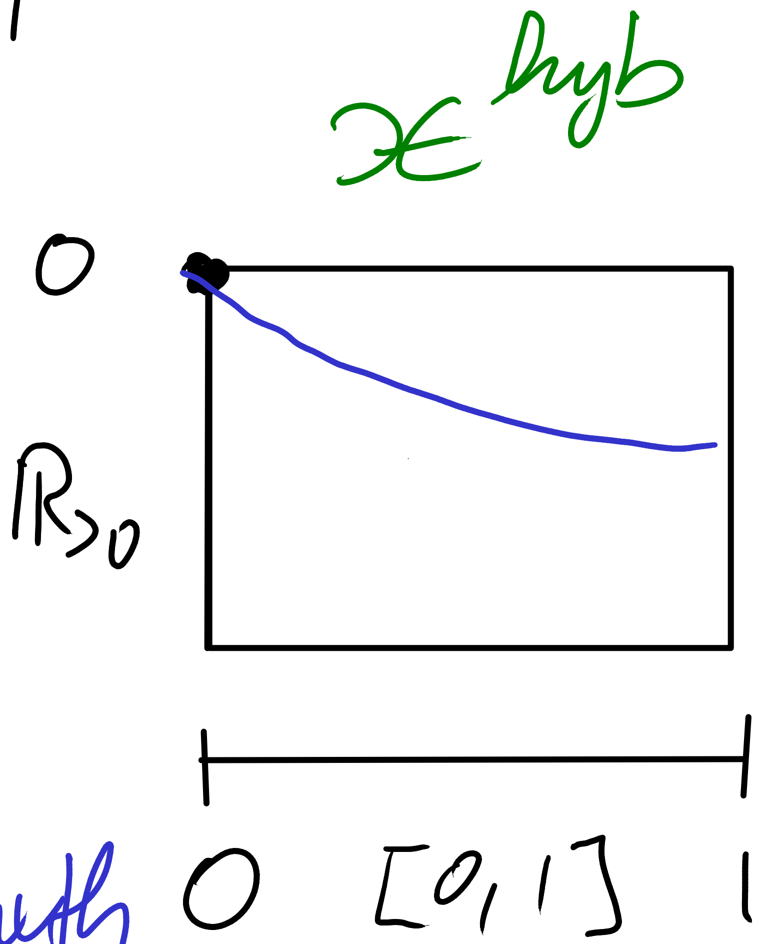


Pol. growth



The hybrid Topology

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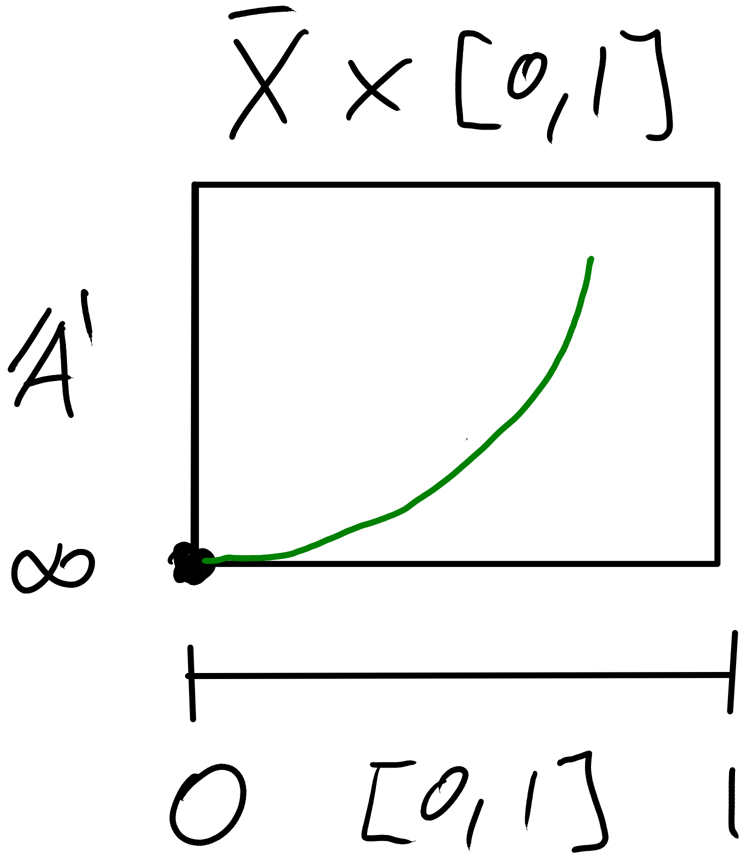
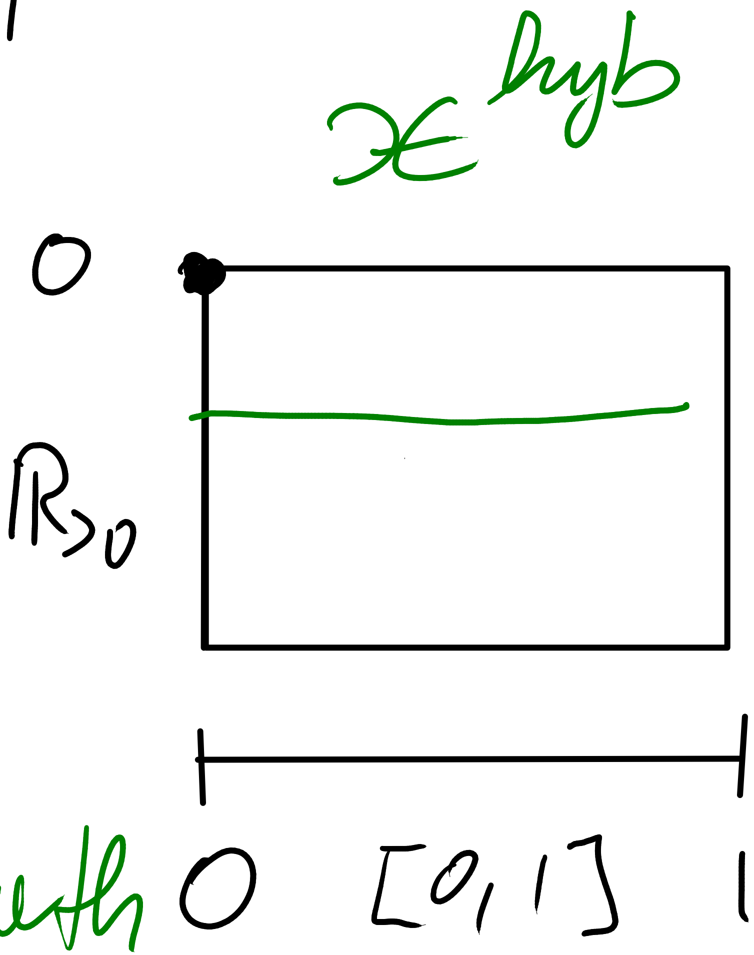


Exp. growth



Results

Recall

$$A^{string} = \sum_g N(g,\alpha') \int_{\mathcal{M}_{g,n}} e^{i\alpha' \mathcal{F}^{st}} \, d\mu^{st}$$

$$A^{QFT} = \sum_g N(g) \int_{\mathcal{M}_{g,n}^{trop}} e^{i\mathcal{F}^{trop}} \, d\mu^{trop}$$

Define

$$\mathcal{F}^{hyb}(\alpha', x) = \begin{cases} \alpha' \mathcal{F}^{st}(x) & \alpha' \neq 0 \\ \mathcal{F}^{trop}(x) & \alpha' = 0 \end{cases}$$

Results

Theorem (A, B, B, F)

The function F^{hyb} is continuous

Results

Theorem (A, B, B, F)

The function $\mathcal{F}^{hyb}$ is continuous

Note: This result generalizes many on the degeneration of the Arakelov Green function and its relation to Zhang's graph Green function

Results

What about the measures?

Results

What about the measures?
local coordinates as before
Consider the family of measures

$$d\mu_{\alpha'} = (\alpha')^{k-h} \frac{dz_1 d\bar{z}_1 \cdots dz_k d\bar{z}_k}{z_1 \bar{z}_1 \cdots z_k \bar{z}_k P(-\log|z_i|)} \omega$$

ω a continuous positive form
P a homogeneous polynomial of degree h

Results

What about the measures?

local coordinates as before

Consider the family of measures

$$d\mu_0 = \int_{D_{\{1,\dots,k\}}} \omega \cdot \frac{dx_1 \cdots dx_k}{P(x_i)}$$

ω a continuous positive form

P a homogeneous polynomial of degree h

Results

What about the measures?

Theorem:

$\{\mu_{\alpha'}, \mu_0\}$ is a continuous family of measures

Results

What about the measures?

Theorem:

$\{\mu_{\alpha'}, \mu_0\}$ is a continuous family of measures

Corollary: If φ is a continuous function on $\mathfrak{X}^{hyb}$ with compact support

$$\lim_{\alpha' \to 0} \int_{\mathcal{M}_{g,n}} e^{i\alpha' \mathcal{F}^{st}} \varphi \, d\mu_{\alpha'} = \int_{\mathcal{M}_{g,n}^{trop}} e^{i\mathcal{F}^{QFT}} \varphi \, d\mu_0$$

Non example

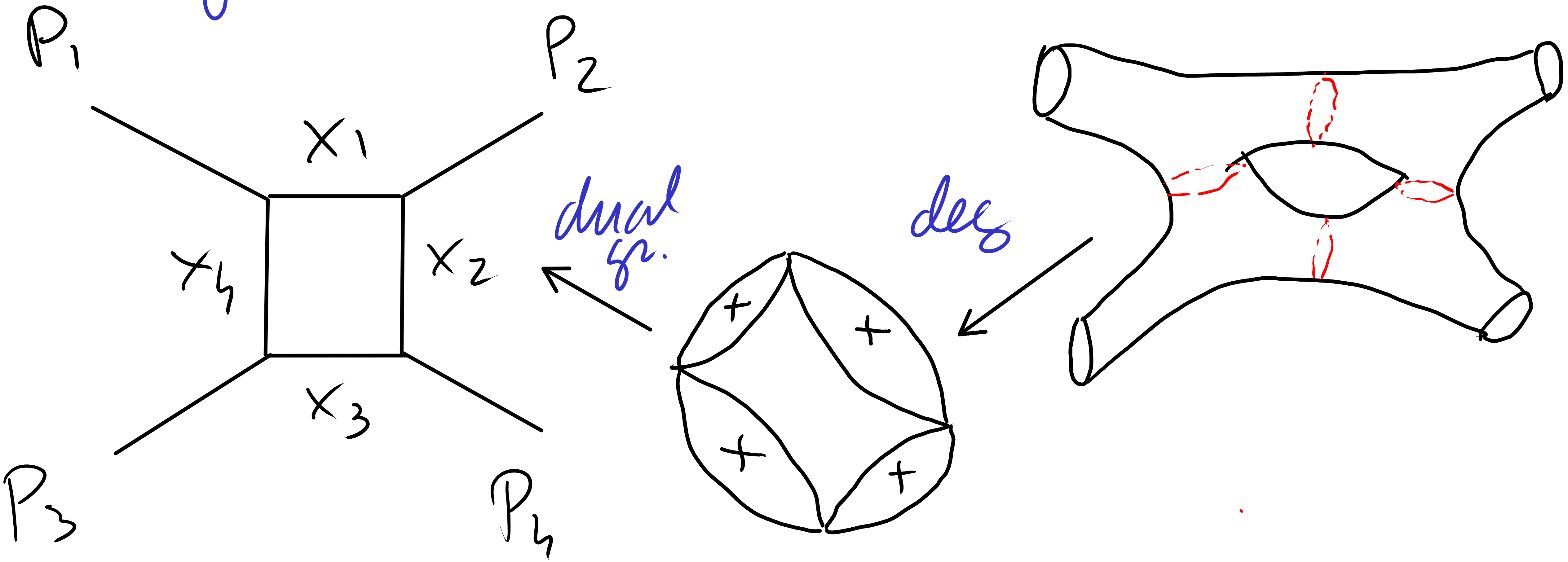
Polyakov measure that appears in bosonic string theory does not fit in this program

$$d\mu_{Pol} \sim \frac{dz\, d\bar{z}}{|z|^4} \quad \text{not integrable} .$$

Example

Fermionic string in 10 dimensions

$g = 1 \qquad n = 4$



Example

Fermionic string in 10 dimensions

Moduli coordinates: $\tau \quad z_2 \; z_3 \; z_4$

moduli var.   elliptic var.

$$d\mu^{\text{string}} = \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \, \frac{d^2z_2 \, d^2z_3 \, d^2z_4}{(\operatorname{Im}\tau)^3}$$

Example

Fermionic string in 10 dimensions

local coordinates: u_1, u_2, u_3, u_4

$$q = e^{2\pi i z} = u_1 \cdot u_2 u_3 u_4$$

$$u_i = e^{2\pi i z_i} \qquad i = 2, \cdots, 3$$

$$d\mu_{\alpha'} = \frac{1}{\alpha'} \frac{d^2u_1\, d^2u_2\, d^2u_3\, d^2u_4}{|u_1|^2 |u_2|^2\, |u_3|^2\, |u_4|^2 \left(\Sigma - \log|u_i|\right)^5}$$

Example

Fermionic string in 10 dimensions

Hence the limit measure is:

$$d\mu_0 = \frac{dx_1\, dx_2\, dx_3\, dx_4}{(\sum x_i)^5}$$

Example

Fermionic string in 10 dimensions

Hence the limit measure is:

$$d\mu_0 = \frac{dx_1\,dx_2\,dx_3\,dx_4}{\left(\sum x_i\right)^5} = \frac{dx_1\,dx_2\,dx_3\,dx_4}{\left(\Psi_{\square}(x_i)\right)^{D/2}}$$

Thank You