Plane models of modular curves

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Modular curves

- The **upper half plane** is $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.
- It admits an action of $\mathrm{GL}_2^+(\mathbb{R})$ by **Möbius transformations**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathfrak{H} \to \mathfrak{H}, \quad z \mapsto \gamma z = \frac{az+b}{cz+d}$$

- For a discrete $\Gamma \leq \mathrm{GL}_2^+(\mathbb{R})$, can form $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$.
- Specific groups Γ of interest

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

- Compactify using **cusps**

$$X(\Gamma) = Y(\Gamma) \cup (\Gamma \backslash \mathbb{P}^1(\mathbb{Q})), \quad X_0(N) = X(\Gamma_0(N))$$

Models for modular curves

Theorem (Shimura (1994))

There exists a smooth projective curve X_Γ over $\mathbb{Q}(\zeta_n)$ such that $X_\Gamma(\mathbb{C}) = X(\Gamma)$. X_Γ is called a **model** *for $X(\Gamma)$.*

Theorem (Galbraith (1996))

There exists an algorithm to compute a model over $\mathbb{Q}$ for $X_0(N)$.

Example (Freitas, Le Hung, and Siksek (2015))

Explicit models for $X_0(15), X_0(35), X_0(75), X_0(225)$ were used to complete the proof of modularity of elliptic curves over real quadratic fields.

Question

When does X_Γ admit a smooth plane model defined over $\mathbb{Q}$?

Reducing to finite computation

Theorem (Anni, A. and García, (2022))

Finitely many modular curves admit a smooth plane model over $\mathbb{Q}$.

Proof.

X_Γ is an orientable compact Riemann surface of genus g. Denote by γ the **gonality** of X_Γ, i.e. the minimum degree of a non-constant map $X_\Gamma \to \mathbb{P}^1$.
Using the Yang-Yau inequality for the first eigenvalue of a compact Riemann surface (Li and Yau (1982)), one bounds the first eigenvalue of the Laplacian on X_Γ by $\lambda_1 < \frac{24\gamma}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma]}$. On the other hand, Selberg's inequality, improved by Kim and Sarnak (2003), yields a lower bound $\lambda_1 \geq \frac{975}{4096}$. For a smooth plane curve of degree d we have $\gamma = d - 1$ and $g = \frac{1}{2}(d-1)(d-2)$. From Gauss-Bonnet we get $g \leq \frac{1}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] + 1$ hence the inequality yields $d \leq 18$. Finally, the number of Γ of a given genus is finite, by Cox and Parry (1984). □

Canonical models

Theorem (Noether-Enriques-Petri)

Let C be a smooth projective curve of genus $g \geq 2$, which is not hyperelliptic. Then the canonical divisor K induces an embedding $\phi_K : C \to \mathbb{P}^{g-1}$, and the ideal defining $\phi_K(C)$ is generated by elements of degree 2, except in the following cases where an element of degree 3 is also needed.

- *$g = 3$, so C is a smooth plane quartic.*
- *$g \geq 4$ and C is a trigonal curve.*
- *$g = 6$ and C is a smooth plane quintic.*

Theorem (Box (2021), Zywina (2020))

Let $G \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be such that $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$, $-1 \in G$ and $\eta G \eta^{-1} = G$, where $\eta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then there exists an algorithm to compute a canonical model over $\mathbb{Q}$ for X_G.

Groups of Shimura type

Problem

Long running time! Polynomial in N, but of high degree.

Solution

1. Compute what we can.
2. Restrict to a family which is easier to compute.

Definition (Group of Shimura type)

Let $H \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ be a subgroup, $t \mid N$, and consider

$$G(H,t) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : a \in H, t \mid b \right\}.$$

Its pullback to $\mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup of **Shimura type**.

Smooth plane models

1. For $d \leq 3$, $g \in \{0, 1\}$, there is always a smooth plane model.
2. For $d = 4$, $g = 3$, so either hyperelliptic or a smooth plane quartic, which is the canonical model.
3. For $d = 5$, if C is a smooth plane quintic, the degree 2 elements of the canonical ideal I_C define a $\mathbb{P}^2$. Evaluating a parametrization at a degree 3 generator recovers the model.
4. In general, we are looking for a **g^2_d-linear series** on C. Write $\phi_K(C) = \operatorname{Proj} S_C$, and consider the minimal free resolution

$$0 \to F_{g-2} \to \ldots \to F_1 \to S \to S_C \to 0$$

Noether proved that F_i is generated in degrees $i + 1$ and $i + 2$. We write $\beta_{i,j}$ for the number of generators of degree j.

Theorem (Green (1984))

If C is a smooth curve that has a g^2_d-linear series, $\beta_{d-4,d-2} \neq 0$.

Results

Congruence subgroups

For $g \leq 24$ (hence $d \leq 8$) Cummins and Pauli (2003) classified all congruence subgroups Γ having such genera.

Theorem (Anni, A. and García, (2022))

There is no modular curve of Shimura type which admits a smooth plane model of degree $d \in \{5, 6, 7\}$. Moreover, a modular curve of Shimura type which admits a smooth plane model of degree 8 must be a twist of one of four curves.

Proof (cases $d = 5, 6$).

For $d = 5$, all have a canonical model generated by quadrics. For $d = 6$, all but one curve have $\beta_{2,4} = 0$. □

Atkin-Lehner involutions

Definition (Atkin-Lehner involution)

For $Q \mid N$ s.t. $(Q, N/Q) = 1$, choose $x, y, z, w \in \mathbb{Z}$ with $y \equiv 1 \bmod Q, x \equiv 1 \bmod N/Q$ and $Qxw - (N/Q)yz = 1$. Then $W_Q = \begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix}$ normalizes $\Gamma_0(N)$, hence induces an **Atkin-Lehner involution** on $X_0(N)$. If W_Q normalizes $\Gamma \subseteq \Gamma_0(N)$, it also induces an involution on X_Γ.

Theorem (Harui, Kato, Komeda, and Ohbuchi (2010))

An involution on a smooth plane curve of degree d has $d + \frac{1-(-1)^d}{2}$ fixed points, and the quotient has gonality $[d/2]$.

Finishing the proof

Proof (cont.)

For $d \in \{7, 8\}$ computing $\beta_{d-4,d-2}$ is beyond us.
But we can look at Atkin-Lehner quotients.
For $d = 7$ all but 6 curves are a degree 4 cover of a hyperelliptic Atkin-Lehner quotient, giving a degree 8 map to $\mathbb{P}^1$, which is impossible by (Greco and Raciti, 1991). For the rest, we use Riemann-Hurwitz to get

$$2g_X - 2 = 2(2g_{X/\langle w \rangle} - 2) + \#X^w$$

for any involution w. Since $g_X = 15$, and for smooth plane curves $\#X^w = 8$, we get $g_{X/\langle w \rangle} = 6$. We find for each curve an AL involution such that the quotient has $g \neq 6$.
This method also works for $d = 8$ for all but 5 curves. One can use the Betti numbers of the quotient to rule out $X_0(256)$ as well. □

A trigonal superelliptic modular curve

- We also computed models for groups not of Shimura type.
- Among the curves of genus 6 we have found one (18A6) canonical model which is not generated by quadrics.
- This yields a trigonal superelliptic modular curve, with the equation

$$y^3 = (x-3)(x+1)(x^2+3)(x+3)^2(x^2+6x+21)^2$$

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