On accumulation and complexity of rational points in projective varieties

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Outline

- Recall about Diophantine arithmetic geometry, projective varieties and fields of definition.
- Recall about canonical divisors for nonsingular projective varieties.
- Some conjectures about existence and distribution of integral and rational points.
- Geometry of numbers and Schmidt's Subspace Theorem.
- Local Weil and Height functions.
- Vojta's Main Conjecture.
- Influence of toric geometry, Convex (Newton-Okounkov) bodies for big linear series, DH-measure and differentiability of the volume function.
- $\mathbb{K}$-stability for $\mathbb{Q}$-Fano varieties and Vojta's Main Conjecture.
- Additional recent results and progress.

Diophantine arithmetic geometry

- **Main Goal.** Study the solutions of those algebraic equations, which are defined over algebraic number fields and/or rings of algebraic integers.
- **Tools and Challenges.** The underlying arithmetic, algebraic and birational geometry of Diophantine equations.
- **Key guiding questions.** How to measure arithmetic closeness and complexity of rational points and solutions to Diophantine arithmetic equations.
- **Influence from birational geometry.** Distribution and complexity of rational points, in projective varieties, should be measured along rational curves; further the Kodaira dimension of a given birational equivalence class should play a role.

Recall about Projective Space

- Let $\mathbf{K} \subseteq \mathbb{C}$ be a number field.
- Projective n-space over $\mathbf{K}$ is defined to be:

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbf{K}} = \{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}_{\mathbf{K}} \setminus \{0\}\}/\sim,$$

  where

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

  if and only if $x_i = \lambda y_i$ for each i and some $0 \neq \lambda \in \mathbf{K}$.
- $\mathbb{P}^n_{\mathbf{K}}$ is a basic example of a moduli space:

  $\mathbb{P}^n = \mathbb{P}(V) = \{$1-dim'l quotients of an $n+1$ dim'l v.sp. $V\}$.
- $\mathbb{P}^n$ is covered by affine spaces $\mathbb{A}^n_{\mathbf{K}}$:

$$U_i = \{z = [z_0 : \cdots : z_n] \in \mathbb{P}^n : z_i \neq 0\}, \ i = 0, \ldots, n.$$

  Then $\mathbb{P}^n = \bigcup_i U_i$ and $\phi_i : U_i \xrightarrow{\sim} \mathbb{A}^n_{\mathbf{K}}$ via:

$$z = [z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right).$$

Recall about Projective Varieties

- Are irreducible and reduced Zariski closed subsets
$$X \subseteq \mathbb{P}^n_{\mathbf{K}},$$
which are defined by the condition that:
$$X = \mathbb{V}(I) = \{(z_0, \dots, z_n) \in \mathbb{P}^n_{\mathbf{K}} : \\ F_1(z_0, \dots, z_n) = \cdots = F_\ell(z_0, \dots, z_n) = 0\}$$
for homogeneous polynomials $F_i(z_0, \dots, z_n)$ generating a homogeneous prime ideal
$$I = \langle F_1, \dots, F_\ell \rangle \subseteq \mathbf{K}[z_0, \dots, z_n].$$
- **Homogeneous Ideal Variety Correspondence:**
prime homogeneous ideals $I \subsetneq \langle z_0, \dots, z_n \rangle$ in $\overline{\mathbf{K}}[z_0, \dots, z_n]$
$$\underset{\text{``}\mathbb{I}\text{''}}{\overset{\text{``}\mathbb{V}\text{''}}{\rightleftarrows}}$$
non-empty varieties in $\mathbb{P}^n$: $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Recall about canonical divisors for nonsingular projective varieties

- Let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety with sheaf of differentials
$$\Omega_X = \Omega_{X/\mathbf{K}}.$$
- Recall, that Ω_X is a locally free $\mathcal{O}_X$-module and is equipped with a universal $\mathbf{K}$-derivation
$$\mathrm{d}\colon \mathcal{O}_X \to \Omega_X.$$
- The canonical line bundle of X is the invertible sheaf
$$\mathrm{K}_X = \bigwedge^{\dim X} \Omega_X.$$
- By a slight abuse of terminology, we also say that K_X is a canonical divisor.

Recall about ample and very ample line bundles

- Let L be a line bundle on a nonsingular projective variety X.
- Recall, that morphisms from X to $\mathbb{P}^n$ are determined by base point free linear systems $|V|$, for

$$0 \neq V \subseteq \mathrm{H}^0(X, L),$$

  $n = \dim V - 1$.
- L is called very ample if the complete linear system $|\mathrm{H}^0(X, L)|$ determines an embedding of X into $\mathbb{P}^n$, $n = h^0(X, L) - 1$.
- L is called ample if $L^{\otimes m}$ is very ample for some $m > 0$.

Recall about big line bundles

- Let L be a line bundle on a nonsingular projective variety X. Then, L is called big if any (and actually all) of the following conditions holds true:
  1. There exists a constant $C > 0$, which is such that
  $$h^0(X, L^{\otimes m}) \geqslant Cm^{\dim X},$$
  for all sufficiently large positive integers $m > 0$.
  2. Denoting by $\kappa(X, L)$ the Iitaka dimension of L, it holds true that
  $$\kappa(X, L) = \dim X.$$
  3. The volume of L:
  $$\mathrm{Vol}(L) := \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^{\dim X}/\dim X!}$$
  is nonzero.
  4. For each ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor E which is such that
  $$L^{\otimes m} \simeq \mathcal{O}_X(A + E).$$

Some conjectures for existence, distribution and accumulation of rational points

- **Conj. (Weak Lang Conj.)** Let X be a general type projective variety defined over a number field $\mathbf{K}$. Then, its set of $\mathbf{K}$-rational points $X(\mathbf{K})$ is not Zariski dense.
- **Conj. (Harris and Tschinkel)** Let X be a nonsingular projective variety defined over a number field $\mathbf{K}$. If its anticanonical bundle $-\mathrm{K}_X$ is numerically effective, then for some finite extension field $\mathbf{F}/\mathbf{K}$, its set of $\mathbf{F}$-rational points $X(\mathbf{F})$ is Zariski dense.
- **Conj. (D. McKinnon)** If $x \in X(\overline{\mathbf{K}})$ is an algebraic point in a polarized projective variety (X, L), defined over a number field $\mathbf{K}$, and if $x \in C$, for some $\mathbf{K}$-rational curve $C \subseteq X$, then x admits a sequence of best approximation with respect to L; such an approximating sequence may be chosen to lie along some rational curve of best approximation in X and through x.

Motivational comments about Schmidt's Subspace Theorem

- Schmidt's Subspace Theorem has emerged as a key tool for studying rational and integral points in projective varieties. (Especially following the program of Corvaja-Zannier.)
- Geometry of numbers, successive minima and Minkowski's second convex body theorem play a key role in its proof.
- In recent times, a good deal of attention has been given to geometric and extended general formulations of the Subspace Theorem.
- For instance, the Subspace Theorem implies General Diophantine Arithmetic Inequalities for projective varieties. (This is the work of Ru-Vojta.)
- In turn, such inequalities can be used to deduce instances of Vojta's Main Conjecture. There is interplay with the area of K-stability for projective varieties.

Motivational comments about influence of higher dimensional birational geometry

- An important mechanism that connects all of these seemingly disjoint topics is:
  - the theory of Newton-Okounkov bodies;
  - the theory of the Duistermaat-Heckman measures; and
  - toric geometry quite generally.
- In what follows, we want to state a classical form of the Subspace Theorem, give a hint a some of its geometric applications and explain its relation, for example, to Vojta's Main Conjecture.

Recall about absolute values

- Suppose that **K** is a number field of degree

$$r_1 + 2r_2 = [\mathbf{K} : \mathbb{Q}].$$

- Then **K** has r_1 real embeddings and r_2 pairs of complex conjugate embeddings.
- There are two kinds of absolute values on **K** which extend the usual and p-adic absolute values on $\mathbb{Q}$.
- Such absolute values are classified as being either Archimedean or non-Archimedean.
- The Archimedean places correspond to embeddings $\sigma : \mathbf{K} \hookrightarrow \mathbb{C}$; complex conjugate embeddings are identified.
- The non-Archimedean places correspond to prime ideals in the ring of integers of **K**.

Recall about product formula

- $M_{\mathbb{Q}} := \{|\cdot|_p : \ p \text{ a prime number or } p = \infty\}$.
- $|\cdot|_\infty$ the usual absolute value on $\mathbb{Q}$.
- If p is a prime number, then $|p|_p = \frac{1}{p}$.
- For a number field $\mathbf{K}$, $M_{\mathbf{K}} := \{|\cdot|_v : v \text{ is a place of } \mathbf{K}\}$.
- $|\cdot|_v := |N_{\mathbf{K}_v/\mathbb{Q}_p}(\cdot)|_p^{1/[\mathbf{K}:\mathbb{Q}]}$ if $v \mid p$, for $p \in M_{\mathbb{Q}}$.
- **Thm. (See e.g., [BG, Prop. 1.4.4]).** Let $\mathbf{K}$ be a number field. The set $M_{\mathbf{K}}$ satisfies the product formula:

$$\prod_{v \in M_{\mathbf{K}}} |x|_v = 1 \text{ for all } x \in \mathbf{K} \setminus \{0\}.$$

- **Sketch of Proof.** WLOG, $\mathbf{K} = \mathbb{Q}$ and x is a prime number. Then

$$\prod_{p \in M_{\mathbb{Q}}} |x|_p = |x|_x |x|_\infty = \frac{1}{x} x = 1.$$

Subspace Theorem set-up

- Let $\mathbf{K}$ be a number field with set of places $M_{\mathbf{K}}$. The multiplicative projective height of

$$x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbf{K})$$

is defined to be

$$H_{\mathcal{O}_{\mathbb{P}^n}(1)}(x) = H(x) := \prod_{v \in M_{\mathbf{K}}} ||x||_v = \prod_{v \in M_{\mathbf{K}}} \max_{0 \leqslant i \leqslant n} |x_i|_v.$$

It is well defined because of the product formula.

- Let S be a finite subset of $M_{\mathbf{K}}$. For each $v \in S$, let

$$\ell_{v0}(x), \ldots, \ell_{vn}(x) \in \mathbf{K}_v[x_0, \ldots, x_n]$$

be a collection of $\mathbf{K}$-algebraic linearly independent linear forms.

Subspace Theorem (Multiplicative Projective formulation)

- **Thm. (See e.g., [BG, Thm. 7.2.2]).** If $\epsilon > 0$, then the set of solutions $x \in \mathbb{P}^n(\mathbf{K})$ of the inequality

$$\prod_{v \in S} \prod_{i=0}^{n} \frac{|\ell_{vi}(x)|_v}{||x||_v} < H(x)^{-n-1-\epsilon}$$

lies in a finite union $T_1 \bigcup \cdots \bigcup T_h$ of proper linear subspaces of $\mathbb{P}^n$.

- **Example.** Lang's formulation of Roth's Theorem, see e.g., [BG, Thm. 6.2.3], follows from the Subspace Theorem. The idea is to contemplate consequences of the Subspace Theorem, when applied to the binary linear forms

$$\ell_{v0}(x) = x_0, \ell_{v1}(x) = x_1 - \alpha_v x_0 \in \mathbf{K}_v[x_0, x_1],$$

for $v \in S$.

Selected guiding questions for Schmidt's Subspace theorem

- As emphasized by Evertse and Schlickewei, the main guiding questions continue to be
  - to algorithmically determine all solutions;
  - to give an upper bound for the number of solutions;
  - to determine the linear scattering of the Diophantine exceptional set; and
  - to establish generalizations.
- Selected recent results and progress:
  - Vojta's Main Conjecture and K-unstable Fano varieties.
  - Roth type inequalities and uniform arithmetic K-instability for polarized klt pairs (X, Δ).
  - Harder and Narasimhan data and central limit theorem for filtered vector spaces.
  - A (Parametric) Subspace Theorem, for linear systems with respect to twisted height functions and linear scattering of Diophantine exceptional sets.
  - Compactness of Diophantine approximation sets.

Twisted height functions

- The concept of twisted height function arose in work of Roy-Thunder, Evertse-Schlickewei and Evertse-Ferretti.
- Let $c_{vi} \in \mathbb{R}$, for $v \in S$, and $i = 0, \ldots, n$, be such that

$$\sum_{i=0}^{n} c_{vi} = 0, \text{ for } v \in S.$$

- For $Q \geqslant 1$, the twisted height function is defined by

$$\begin{aligned} H_Q(x) &:= \prod_{v \in S} \left(\max_{0 \leqslant i \leqslant n} |\ell_{vi}(x)|_v Q^{-c_{vi}} \right) \cdot \prod_{v \notin S} ||x||_v \\ &= \prod_{v \in S} \left(\max_{0 \leqslant i \leqslant n} \frac{|\ell_{vi}(x)|_v}{||x||_v} Q^{-c_{vi}} \right) \cdot H(x). \end{aligned}$$

Subspace Theorem (Parametric formulation)

- **Rmk.** These (equivalent) projective and affine forms of the Subspace Theorem are implied by the Parametric Subspace Theorem. The parametric formulation, which was given by Evertse-Ferretti-Schlickewei involves the twisted height functions.
- **Thm. (Evertse-Ferretti-Schlickewei).** Let $\delta > 0$. Then, there exists a real number $Q_0 > 1$ and a finite number of proper linear subspaces $T_1, \dots, T_h \subsetneq \mathbb{P}^n$ such that for all $Q \geqslant Q_0$, there is a $T_i \in \{T_1, \dots, T_h\}$ with the property that

$$\left\{x \in \mathbb{P}^n(\mathbf{K}) : H_Q(x) \leqslant Q^{-\delta}\right\} \subseteq T_i.$$

- **Thm. (-).** Parametric subspace thm for twisted height functions and linear systems $\Rightarrow$ FW-type inequalities for linear systems $\Rightarrow$ Subspace Thm. for linear systems.

Preliminaries for Vojta's Main Conjecture

- Let X be a projective variety defined over a number field $\mathbf{K}$ and D a Cartier divisor on X and defined over some finite extension of $\mathbf{K}$. Consider the proximity function

$$m_S(\cdot, D) := \sum_{v \in S} \lambda_D(\cdot, v)$$

for D with respect to a finite set $S \subseteq M_{\mathbf{K}}$ of places of $\mathbf{K}$.

- Here, the local Weil functions $\lambda_D(\cdot, v)$ are described as:

$$\lambda_D(x, v) = -\log(v\text{-adic distance from } x \text{ to } D).$$

- The logarithmic height functions determined by very ample line bundles L on X are described by:

$$h_L(x) = \sum_{v \in M_{\mathbf{K}}} \max_j \log |x_j|_v.$$

- In general, the height function of an arbitrary line bundle M on X, (defined over $\mathbf{K}$) is obtained by first expressing M as the difference of two ample line bundles.

Vojta's Main Conjecture

- Let X be a non-singular projective variety defined over a number field $\mathbf{K}$. Let S be a fixed finite set of places of $\mathbf{K}$ and let
$$D = D_1 + \cdots + D_q$$
be a normal crossings divisor on X.
- **Conj. (Vojta).** Let L be a big line bundle on X, defined over $\mathbf{K}$, and let $\epsilon > 0$. Then there exists a proper Zariski closed subset
$$Z \subsetneq X$$
so that for all
$$x \in X(\mathbf{K}) \setminus Z(\mathbf{K})$$
it holds true that
$$m_S(x, D) + h_{\mathrm{K}_X}(x) \leqslant \epsilon h_L(x) + \mathrm{O}(1).$$

Vojta's Main Conjecture: first examples

- **E.g.** For the case that $X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$, and $D = H_0 + \cdots + H_n$, for H_i hyperplanes in general position and then the inequalities given by Vojta's Main Conjecture become those of Schmidt's Subspace Theorem.
- **E.g.** For the case that X is of general type, then Vojta's Main Conjecture together with Northcott's theorem, for finiteness of points of bounded height, implies non-Zariski denseness of the set of $\mathbf{K}$-rational points in X. In particular, Vojta's Main Conjecture implies the Bombieri-Lang conjecture.

Some recent results

- In the direction of Vojta's Main Conjecture, we mention one important consequence of the Arithmetic General Theorem ([RV] and [Gri]).
- First, we need to describe one auxiliary concept which arises in a variety of settings.
- **Defn.** A $\mathbb{Q}$-Fano variety is a projective variety X, which has log terminal singularities and ample $\mathbb{Q}$-Cartier anti-canonical class $-\mathrm{K}_X$.
- **Defn.** If E is a divisor over a $\mathbb{Q}$-Fano variety X, then let $\pi\colon X' \to X$ be a model with $E \subseteq X'$ a Cartier divisor and put:

$$\beta(-\mathrm{K}_X, E) := \int_0^\infty \frac{\mathrm{Vol}(\pi^*(-\mathrm{K}_X) - tE)}{\mathrm{Vol}(-\mathrm{K}_X)} \mathrm{d}t.$$

  This is the expected order of vanishing of $-\mathrm{K}_X$ along E.
- **E.g.** If $X = \mathbb{P}^n$ and E is a hyperplane, then

$$\beta(-\mathrm{K}_X, E) = 1.$$

- **Thm. (-).** Let X be a $\mathbb{Q}$-Fano variety defined over a number field $\mathbf{K}$. Fix a finite set of places $S \subseteq M_{\mathbf{K}}$. Let E be a prime divisor over X and having field of definition some finite extension of $\mathbf{K}$. Assume that $\beta(-\mathrm{K}_X, E) \geqslant 1$. Fix L a big line bundle on X, defined over $\mathbf{K}$, and let $\epsilon > 0$. Then there exists a Zariski closed subset $Z \subsetneq X$ such that if $x \in X(\mathbf{K}) \setminus Z(\mathbf{K})$, then

$$m_S(x, D) + h_{\mathrm{K}_X}(x) \leqslant \epsilon h_L(x) + \mathrm{O}(1).$$

Here $D = D_1 + \cdots + D_q$ is a divisor over X that has the properties that:

 (i) the divisors $D_1, \ldots, D_q$ are each linearly equivalent to E; and
 (ii) the divisors $D_1, \ldots, D_q$ intersect properly.

- **Sketch of Proof.** It suffices to establish the inequality

$$m_S(x, D) \leqslant (\epsilon + 1) h_{-\mathrm{K}_X}(x) + \mathrm{O}(1)$$

for all $x \in X(\mathbf{K}) \setminus Z(\mathbf{K})$ and $Z \subsetneq X$ some proper Zariski closed subset. This is implied by [Gri] and/or [RV].

A first example

- To gain some intuition for the conclusion of the Theorem, consider the following example.
- **E.g.** When $X = \mathbb{P}^n$ and $E \subseteq \mathbb{P}^n$ is a hyperplane, we then have that
$$\beta(-\mathrm{K}_X, E) = 1.$$
The conclusion of the Theorem applied to $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and
$$D = D_1 + \cdots + D_{n+1},$$
for $D_1, \ldots, D_{n+1}$ a collection of hyperplanes in general position, recovers the usual statement of Schmidt's Subspace Theorem.

Influence of Toric Geometry

- The quantities $\beta(-\mathrm{K}_X, E)$ are related to the Duistermaat-Heckman measures and have origins in toric geometry. They have an interpretation via the theory of Okounkov bodies through the concept of concave transforms.
- **E.g.** Consider a toric blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\pi\colon S(\Sigma') = \mathrm{Bl}_{\{pt\}}(\mathbb{P}^1 \times \mathbb{P}^1) \to S(\Sigma) = \mathbb{P}^1 \times \mathbb{P}^1.$$

Our conventions are such that the primitive ray vectors for the respective fans Σ' and Σ are given by:

$$v_0' = (1,1), v_1' = (-1,0), v_2' = (0,1),$$
$$v_3' = (1,0), v_4' = (0,-1)$$

and

$$v_1 = (-1,0), v_2 = (0,1), v_3(1,0), v_4 = (0,-1).$$

- The polytopes of the divisors, for $t \in \mathbb{R}_{\geqslant 0}$,

  $$\pi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) - tE \sim a\pi^* D_3 + b\pi^* D_4 - tE,$$

  where $a, b > 0$ and $a \leqslant b$, are cut out by the inequalities:
  - $(m_1, m_2) \cdot (1, 1) \geqslant -a + t$,
  - $(m_1, m_2) \cdot (-1, 0) \geqslant 0$,
  - $(m_1, m_2) \cdot (0, 1) \geqslant 0$,
  - $(m_1, m_2) \cdot (1, 0) \geqslant -a$,
  - $(m_1, m_2) \cdot (0, -1) \geqslant -b$.

- By determining the areas of these polytopes it follows that if
$$f(t) = \frac{\mathrm{Vol}(a\pi^* D_3 + b\pi^* D_4 - tE)}{\mathrm{Vol}(a\pi^* D_3 + b\pi^* D_4)}$$
then
$$f(t) = \begin{cases} 1 - \frac{t^2}{2ab} & \text{if } 0 \leqslant t \leqslant a; \\ 1 + \frac{a}{2b} - \frac{t}{b} & \text{if } a \leqslant t \leqslant b \\ \frac{(a-b-t)^2}{2ab} & \text{if } b \leqslant t \leqslant a+b. \end{cases}$$
- Finally, by integrating $f(t)$, we obtain that
$$\beta_x(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)) = \beta(L,E) = \int_0^{a+b} f(t)\mathrm{d}t = \frac{a+b}{2}.$$
- **Rmk.** This example helps to give intuition as to the more general statements for calculating expected orders of vanishing via the theory of concave transforms for Okounkov bodies. ([Gri], [BKMS], [BC].)

Influence of K-stability

- As another interesting consequence of the Theorem, we indicate some ideas from K-stability.
- **Valuative criteria of K-stability (K. Fujita and C. Li).** A $\mathbb{Q}$-Fano variety X is not K-stable if and only if

$$\beta(-\mathrm{K}_X, E) \geqslant 1 + a(X, E)$$

  for at least one prime divisor E over X and defined over some finite extension of the base number field. Here, $a(X, E)$ is the discrepancy of E with respect to X.
- This criteria for K-stability together with the Theorem imply the following interesting consequence. It establishes instances of Vojta's Main Conjecture for $\mathbb{Q}$-Fano varieties, that have canonical singularieties, are not K-stable.
- **Cor. (-).** Let X be a $\mathbb{Q}$-Fano variety with canonical singularieties. If X is not K-stable, then the conclusion of the Theorem holds true for at least one prime divisor E over X and having field of definition some finite extension of the base number field.

The case of points of bounded degree

- In general, it remains a non-trivial open problem to obtain sharp height inequalities for points of bounded degree.
- However, there is a conjectural formulation of Schmidt's Theorem, with discriminant term, for points of bounded degree. It is a special case of the strong from of Vojta's Main Conjecture, for points of bounded degree.
- **Conj. (Levin).** Let $\mathbf{K}$ be a number field and S a finite set of places. Let $H_1, \dots, H_q \subseteq \mathbb{P}^n$ be a collection of hyperplanes in general position. Put $H = H_1 + \cdots + H_q$. Fix $d \geqslant 1$ and let $\epsilon > 0$. Then, there exists a proper Zariski closed subset $Z \subsetneq \mathbb{P}^n$ such that

$$m_S(x, H) \leqslant (n + 1 + \epsilon) h_{\mathcal{O}_{\mathbb{P}^n}(1)}(x) + \mathrm{d}_{\mathbf{K}}(x) + \mathrm{O}(1)$$

  for all $x \in \mathbb{P}^n(\overline{\mathbf{K}}) \setminus Z(\overline{\mathbf{K}})$ with $[\mathbf{K}(x) : \mathbf{K}] \leqslant d$.
- Unconditional subspace type results, for points of bounded degree, have been given by Levin.

Schlickewei's Subspace conjecture for points of bounded degree

- Another point of departure, for bounded degree height inequalities, is a conjecture, of Schlickewei.
- **Conj. (Schlickewei).** For each $v \in S$, fix linearly independent linear forms $\ell_{v0}(x), \dots, \ell_{vn}(x)$ in the polynomial ring $\overline{\mathbf{K}}[x_0, \dots, x_n]$. Then there exists a positive constant $\mathrm{c}(n, d) > 0$, which depends only on r and d, which has the following property for each fixed $\delta > 0$. If $Z \subsetneq \mathbb{P}^n(\overline{\mathbf{K}})$ is the set of all $x = [x_0 : \dots : x_n] \in \mathbb{P}^n(\overline{\mathbf{K}})$ which satisfy the conditions that
  - $\sum_{v \in S} \sum_{i=0}^{n} \lambda_{\ell_{vi}, v}(x) > (\mathrm{c}(n, d) + \delta) h_{\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbf{K}}}}(1)}(x) + \mathrm{O}(1)$; and
  - $[\mathbf{K}(x) : \mathbf{K}] \leqslant d$,

  then there exist finitely many proper linear subspaces $\Lambda_1, \dots, \Lambda_h$ in $\mathbb{P}^n_{\overline{\mathbf{K}}}$, each having field of definition with degree at most d over $\mathbf{K}$, and such that Z is contained in their union $\Lambda_1 \bigcup \dots \bigcup \Lambda_h$.

An arithmetic general theorem for points of bounded degree

- **Thm. (-).** Schlickewei's conjecture implies the following for a given geometrically irreducible projective variety X over $\mathbf{K}$. Let $D_1, \dots, D_q$ be nonzero effective Cartier divisors on X and defined over a fixed finite extension field $\mathbf{F}/\mathbf{K}$. Put $D = D_1 + \cdots + D_q$, and assume that these divisors D_i intersect properly. Let L be a big line bundle on X. Then, there exist positive constants $\gamma(d, L, D_i)$ so that if $\epsilon > 0$, then

$$\sum_{i=1}^{q} \gamma(d, L, D_i)^{-1} m_S(x, D_i) \leqslant (1+\epsilon)\, h_L(x) + \mathrm{O}(1)$$

for all algebraic points

$$x \in X(\overline{\mathbf{K}}) \setminus \left(Z(\overline{\mathbf{K}}) \bigcup \mathrm{Bs}(L)(\overline{\mathbf{K}}) \bigcup \mathrm{Supp}(D)(\overline{\mathbf{K}}) \right)$$

with $[\mathbf{K}(x) : \mathbf{K}] \leqslant d$. Here, $Z \subsetneq X$ is contained in a finite union of linear sections $\Lambda_1, \dots, \Lambda_h$, with degree $\leqslant d$.

Arithmetic uniform K-instability and (penultimate) Roth's theorem for klt-pairs

- **Thm. (-).** Let (X, Δ) be a Kawamata log terminal pair defined over a number field $\mathbf{K}$. Let L be an ample line bundle on X and defined over $\mathbf{K}$. Fix a finite set of places $S \subseteq M_{\mathbf{K}}$. For each $v \in S$, let E_v be a prime divisor over X and having field of definition some finite extension field of $\mathbf{K}$. Assume that (X, Δ) is not arithmetically K-stable with respect to L and E_v, for each $v \in S$. Moreover, suppose that $R_v \in \mathbb{R}_{>0}$, for $v \in S$, are destabilizing Roth constants; in particular, the inequality

$$1 < \sum_{v \in S} A(E_v, X, \Delta) < \sum_{v \in S} \beta_{E_v}(L) R_v$$

is valid. Then, there exists a proper Zariski closed subset $W \subsetneq X$, defined over $\mathbf{K}$, and at least one place $v \in S$, so that

$$\alpha_{E_v}(\{x_i\}, L) \geqslant 1/R_v.$$