Dirichlet law for factorization of integers, polynomials and permutations

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Introduction

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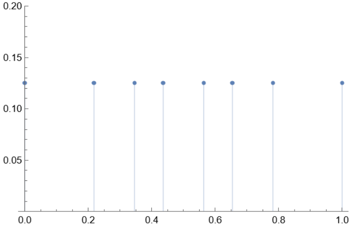

Figure: Probability mass function of D_{24}

DDT arcsine law

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Nevertheless, Deshouillers, Dress and Tenenbaum (DDT) proved the mean of the corresponding distribution functions converges to that of the arcsine law.

Theorem (DDT arcsine law)

Uniformly for $u \in [0,1]$, *we have*

$$\frac{1}{x}\sum_{n\leq x}\mathbb{P}\left(D_n \leq u\right) = \frac{2}{\pi}\arcsin\sqrt{u} + O\left(\frac{1}{\sqrt{\log x}}\right),$$

where $\mathbb{P}\left(D_n \leq u\right) := \frac{1}{\tau(n)}\sum_{\substack{d|n\\ d\leq n^u}} 1$ *is the distribution function of* D_n.

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In particular, it is symmetric and $f(x) \to \infty$ as $x \to 0$ or 1.

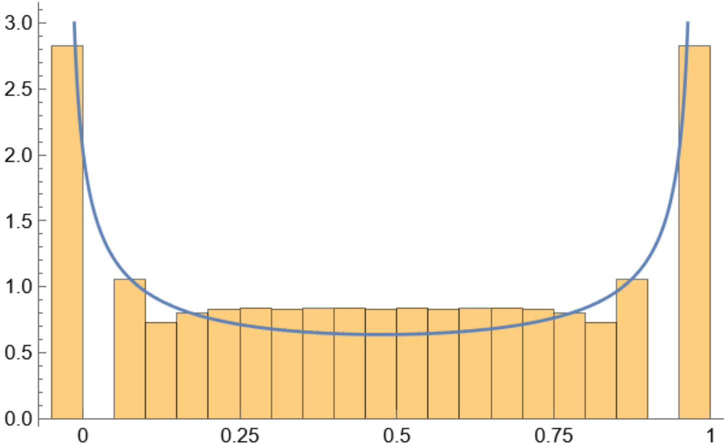
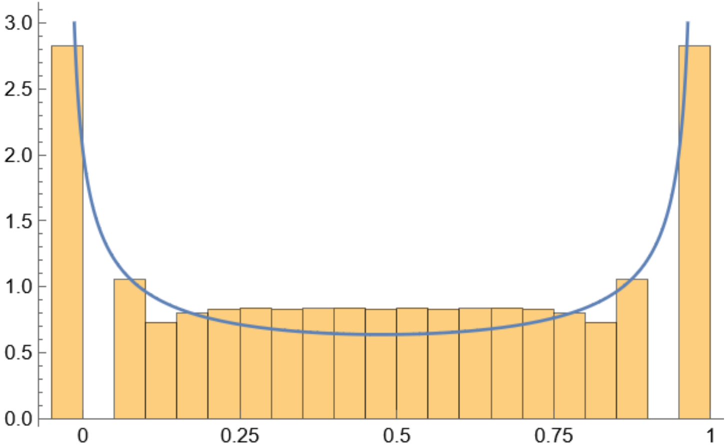


Figure: DDT arcsine law with $x = 10^3$



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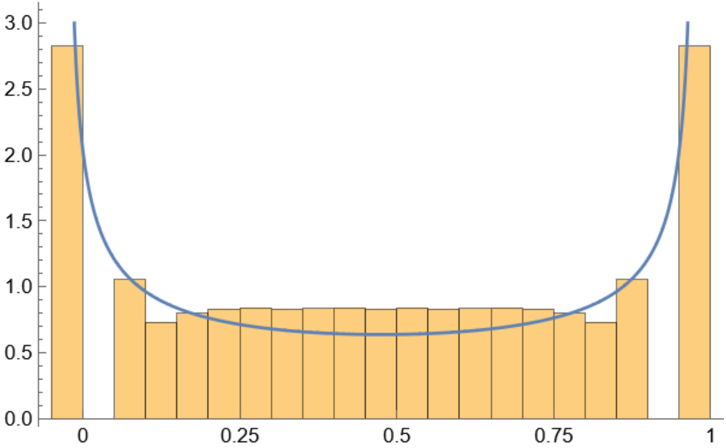

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Motivation. Taking a Bayesian perspective, the arcsine distribution is a special case of Dirichlet distribution. Therefore, a natural question would be how to generalize the DDT theorem?

Dirichlet distribution

Definition

Let $k \geq 2$. The Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_k > 0$ is denoted by $\mathrm{Dir}(\alpha_1, \ldots, \alpha_k)$, which is defined on the $(k-1)$-dimensional probability simplex

$$\{(t_1, \ldots, t_k) \in [0,1]^k \ : \ t_1 + \cdots + t_k = 1\}$$

having density

$$f_{\boldsymbol{\alpha}}(t_1, \ldots, t_k) := \frac{\Gamma\left(\sum_{i=1}^k \alpha_i\right)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k t_i^{\alpha_i - 1}.$$

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In particular, if $\alpha_1, \ldots, \alpha_k < 1$, then $f_{\boldsymbol{\alpha}}(t_1, \ldots, t_k) \to \infty$ rapidly as $t_j \to 1$ for some $j = 1, \ldots, k$.

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Example

When $k = 2, \alpha = \beta = \frac{1}{2}$, Dirichlet distribution reduce to the arcsine distribution.

Main theorem

Theorem (L., 2022)

Let $k \geq 2$ be an integer. Then uniformly for $x \geq 2$ and $u_1, \ldots, u_{k-1} \geq 0$ satisfying $u_1 + \cdots + u_{k-1} \leq 1$, we have

$$\frac{1}{x} \sum_{n \leq x} \frac{1}{\tau_k(n)} \sum_{d_1 \leq n^{u_1}} \cdots \sum_{\substack{d_{k-1} \leq n^{u_{k-1}} \\ d_1 \cdots d_{k-1} | n}} 1 = F_{\mathbf{1/k}}(u_1, \ldots, u_{k-1}) + O_k\left(\frac{1}{(\log x)^{\frac{1}{k}}}\right),$$

where $F_{\mathbf{1/k}}$ is the distribution function of $Dir\left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$.

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where $F_{\mathbf{1/k}}$ is the distribution function of Dir $\left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$.

Remark. The error term here is optimal if full uniformity in $u_1, \ldots, u_{k-1}$ is required.

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$$\left(\frac{\log d_1}{\log n}, \ldots, \frac{\log d_k}{\log n}\right) \xrightarrow{d} Dir\left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$$

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It is a general phenomenon that the "anatomy" of polynomials or permutations is essentially the same as that of integers, and the main theorem here is no exception.

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For polynomials over finite fields, it is similar.

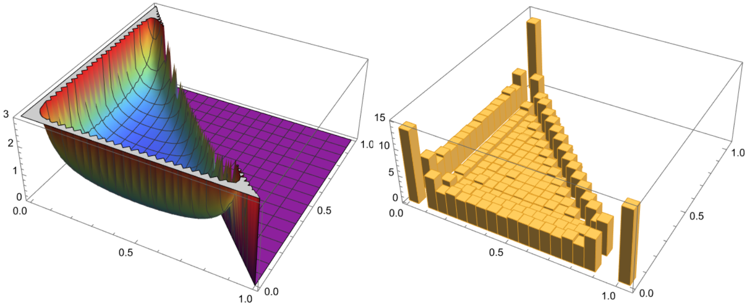
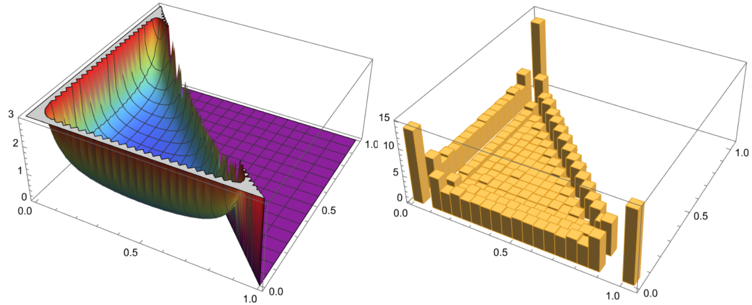


Figure: Main theorem for $k = 3$ with $x = 10^3$



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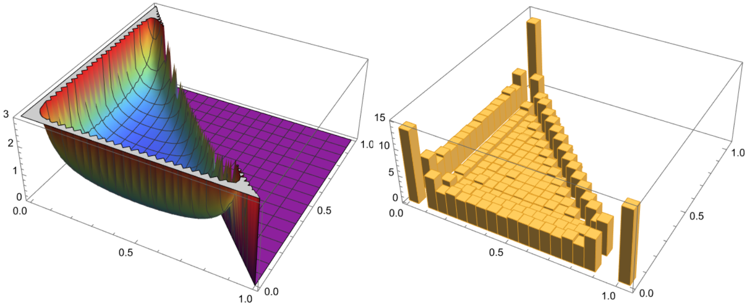

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Since for each i the parameter $\alpha_i = \frac{1}{k}$ is less than 1, the density $f_\alpha(t_1, \ldots, t_k)$ is concentrated on the vertices of the probability simplex. Therefore, our intuition that a typical factorization of integers into k parts consists of $k - 1$ small factors and one large factor is justified quantitatively.

A multiple Dirichlet series

Definition

For $\mathrm{Re}(s_j) > 1, j = 1, \ldots, k$, we denote by $\mathcal{D}(s_1, \ldots, s_k)$ the multiple Dirichlet series

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{\tau_k(n_1 \cdots n_k)^{-1}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

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Although τ_k is not completely multiplicative, we still have

$$\mathcal{D}(s_1, \ldots, s_k) \text{``} \approx \text{''} \zeta(s_1)^{\frac{1}{k}} \cdots \zeta(s_k)^{\frac{1}{k}}.$$

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In particular, it can be expressed as an Euler product and continued meromorphically up to the non-trivial zeros.

Also, since $\zeta(s) \sim \frac{1}{s-1}$ for $s \sim 1$, we have

$$\mathcal{D}(s_1, \ldots, s_k) \sim (s_1 - 1)^{-1} \cdots (s_k - 1)^{-1}$$

for $s_1, \ldots, s_k \sim 1$.

Sketch of proof

First of all, we have

$$\sum_{n\le x}\frac{1}{\tau_k(n)}\sum_{\substack{d_1\le n^{u_1}\\ d_1\cdots d_{k-1}|n}}\cdots\sum_{d_{k-1}\le n^{u_{k-1}}}1$$

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Let $d_k = n/d_1\cdots d_{k-1}$. Then (2.1) becomes

$$\sum_{d_1\leq x^{u_1}}\cdots\sum_{d_{k-1}\leq x^{u_{k-1}}}\sum_{d_k\leq x/d_1\cdots d_{k-1}}\frac{1}{\tau_k(d_1\cdots d_k)}. \tag{2.2}$$

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As we can see, the sum is not symmetric.

Lemma

Let $S(x_1, \dots, x_k)$ denote the weighted sum

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which is $\approx F_{\mathbf{1/k}}(u_1, \ldots, u_{k-1})x$ using the change of variables $x_j = x^{t_j}, j = 1, \ldots, k-1$. It remains to prove the lemma.

Applying Mellin's inversion formula, the weighted sum $S(x_1, \dots, x_k)$ becomes

$$\frac{1}{(2\pi i)^k} \int_{\mathrm{Re}(s_1)=1+\frac{1}{\log x_1}} \cdots \int_{\mathrm{Re}(s_k)=1+\frac{1}{\log x_k}} \left(\frac{\partial^{2k}}{\partial s_1^2 \cdots \partial s_k^2} \mathcal{D}(s_1, \dots, s_k) \right) x_1^{s_1} \cdots x_k^{s_k} \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k},$$

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which is by Cauchy's estimate combined with the classical zero-free region

$$\approx \frac{1}{(2\pi i)^k} \int_{I_1^{(1)}} \cdots \int_{I_k^{(1)}} \left(\frac{\partial^{2k}}{\partial s_1^2 \cdots \partial s_k^2} \mathcal{D}(s_1, \ldots, s_k) \right) \cdot x_1^{s_1} \cdots x_k^{s_k} \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k}, \tag{2.3}$$

where $I_j^{(1)} := \{s_j \in \mathbb{C} : \mathrm{Re}(s_j) = 1 + \frac{1}{\log x_j}, |\mathrm{Im}(s_j)| \leq 1\}$ for $j = 1, \ldots, k$.

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$$\frac{\partial^{2k}}{\partial s_1^2 \cdots \partial s_k^2} \mathcal{D}(s_1, \ldots, s_k) \sim \left(1 + \frac{1}{k}\right)^k \frac{1}{k^k} \cdot (s_1 - 1)^{-\frac{1}{k} - 2} \cdots (s_k - 1)^{-\frac{1}{k} - 2}.$$

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Therefore, the expression (2.3) is

$$\approx \left(1 + \frac{1}{k}\right)^k \frac{1}{k^k} \prod_{j=1}^{k} \left(\frac{1}{2\pi i} \int_{I_j^{(1)}} (s_j - 1)^{-\frac{1}{k}-2} x_j^{s_j} \frac{ds_j}{s_j} \right),$$

Recall that for $s_1, \ldots, s_k \sim 1$, we have

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Let $x > 1, \sigma > 1$ *and* $\mathrm{Re}(\alpha) > 1$. *Then we have*

$$\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} \frac{x^s}{s(s-1)^\alpha} ds = \frac{1}{\Gamma(\alpha)} \int_1^x (\log y)^{\alpha-1} dy.$$

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Remark. The classical contour deformation fails as one of the x_j's can be as small as 1 if $u_j = 0$ so that the contribution of the contour away from the branch point $s_j = 1$ is no longer negligible.

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Remark. The classical contour deformation fails as one of the x_j's can be as small as 1 if $u_j = 0$ so that the contribution of the contour away from the branch point $s_j = 1$ is no longer negligible. Instead, we follow the approach by Granville and Koukoulopoulos to break each contour into three pieces, and the main contribution comes from $|\mathrm{Im}(s_j)| \leq 1$, i.e. close to the branch point $s_j = 1$.

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$$\left(\frac{\log d_1}{\log n}, \ldots, \frac{\log d_k}{\log n} \right) \xrightarrow{d} \mathrm{Dir}\left(\frac{1}{2k}, \ldots, \frac{1}{2k} \right)$$

as $x \to \infty$.

Thank you for listening!