A unified construction of many conjectural rigid Calabi-Yau threefolds

Adam Logan (TIMC/CSE and Carleton University)

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If you missed this talk at the conference, I will be giving a longer version in the Hannover seminar on October 20 and in the Ottawa seminar on November 2. See `researchseminars.org` for details on the latter.

Table of Contents

Elliptic curves

The basic modularity statement was conjectured by Shimura and Taniyama and proved by Wiles and others:

Theorem

Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. For all $p \nmid N$, let $a_p(E) = p + 1 - \#E_{\mathbb{F}_p}$. Then there exists a Hecke eigenform f of weight 2 for $\Gamma_0(N)$ with Hecke eigenvalues $a_p(f) = a_p(E)$.

Since then, this statement has been generalized in all sorts of ways: elliptic curves over number fields, certain abelian varieties, some other higher-dimensional varieties, etc. Much of this is formidably technical.

Eichler-Shimura

The converse to the statement on the last slide, though proved much longer ago, has been much harder to generalize:

Theorem

(Eichler-Shimura) Let f be a Hecke eigenform of weight 2, new at level N, with Hecke eigenvalues $a_p(f)$. Suppose that all $a_p(f) \in \mathbb{Q}$. Then there is an elliptic curve E_f with $a_p(E_f) = a_p(f)$ for all $p \nmid N$.

Proof.

(sketch) Let E be the connected component of 0 in the intersection of the kernels of $T_p - a_p(f)$ on $J_0(N)$. The hard part is to prove that $\dim E = 1$. □

Elkies-Schütt theorem

Elkies and Schütt showed that K3 surfaces are what is needed for $k = 3$:

Definition

A *K3 surface* is a surface S with $K_S = 0, \pi_1(S, \mathbb{Z}) = 0$.

Examples of K3 surfaces include surfaces of degree 4 in $\mathbb{P}^3$, intersections of a quadric and a cubic in $\mathbb{P}^4$, and one type of surface of degree $2d$ in $\mathbb{P}^{d+1}$ for every $d > 0$.

Theorem

(Elkies-Schütt) Let f be a known rational Hecke eigenform of weight 3 and level N. Then there is a K3 surface $S/\mathbb{Q}$ of Picard number 20 with $p^2 + a_p(f) + c(p)p + 1$ points mod p for all $p \nmid N$.

(All such forms are "known" with at most one exception, which is excluded by GRH.)

What about higher weight?

We want to look at varieties that have $\mathrm{h}^{3,0} = \mathrm{h}^{0,3} = 1$ (Hodge decomposition of cohomology over $\mathbb{C}$; or over an arbitrary field we can define $\mathrm{H}^{i,j} = \mathrm{H}^q(X, \Omega^p)$, where Ω^p is the sheaf of holomorphic p-forms). We also want $\mathrm{h}^{i,0} = 0$ for $0 < i < d$, since otherwise the point count mod p does not come only from $\mathrm{h}^{3,0}$. In other words, we want $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.

Definition

A *Calabi-Yau variety* is a smooth variety V of dimension d with K_V trivial and $\mathrm{h}^{i,0} = 0$ for $0 < i < d$.

In dimensions $1, 2$, these are elliptic curves and K3 surfaces.

In dimension 3, they are quintics in $\mathbb{P}^4$, and $(2,4)$ or $(3,3)$ complete intersections in $\mathbb{P}^5$, and ... hundreds of millions of other families.

A big question

Gouvêa and Yui showed that if V is a Calabi-Yau threefold with $\mathrm{h}^{2,1} = 0$, then the traces of Frobenius on $\mathrm{H}^3(V)$ are the eigenvalues of a Hecke eigenform of weight 4.

So one might ask (and in fact Mazur and van Straten have asked):

Question

For which rational Hecke eigenforms f of weight 4 is there a Calabi-Yau threefold with $\mathrm{h}^{2,1} = 0$ and H^3 described by f?

Example

Example

Let X be the double cover of $\mathbb{P}^3$ defined by $t^2 = xyzw(x+y)(y+z)(z+w)(w+x)$. Then there is a Calabi-Yau resolution of X with $\mathrm{h}^{2,1} = 0$, corresponding to the cusp form of weight 4 and level 8.

There are about 10 other arrangements of 8 planes with the same property, but they give only 4 different modular forms.

It is not a coincidence that there is no deformation of the set of 8 planes that preserves all the intersections.

What is known so far

Several dozen forms of weight 4 have been realized by rigid Calabi-Yau threefolds, and about the same number by nonrigid ones.

In addition to double cover constructions as on the last slide, there are various other sources: products of elliptic surfaces, mirror symmetry, hypergeometric families, quantum field theory, ...

The new construction

The goal of this talk is to describe a new construction that I have discovered that allows for the realization of a lot of previously known and new weight-4 cusp forms.

The construction has the advantage of being simple and uniform, and allowing the level of the cusp form attached to the threefold to be predicted in many cases.

The disadvantage is that it seems quite difficult to prove that the varieties I construct are actually rigid Calabi-Yau threefolds. In principle this can be proved by a finite computation, but that computation is not one that anyone would want to do.

Also, the construction is inherently bounded: it can only work for finitely many forms up to twist. Even if there are only finitely many forms up to twist, we are very far short of realizing them all.

Table of Contents

The Picard number

Let S be a smooth surface. Given two distinct curves on S, one can define the *intersection number*, and in fact one can define the *self-intersection number* of a curve.

This gives a symmetric bilinear form on the free abelian group of curves. The quotient by the kernel is called the *Néron-Severi group* of S. The symmetric bilinear form is still defined there.

The rank of this group is the *Picard number* of S.

If we fix a rank r and a bilinear form, there is a moduli space of K3 surfaces with Néron-Severi group $\mathbb{Z}^r$ and the given form. Its dimension is $20 - r$.

Families of K3 surfaces

Let $T \to \mathbb{P}^1$ be a family of K3 surfaces over $\mathbb{Q}$.

If there is a two-dimensional moduli space of K3 surfaces like the fibres, then we can deform to another family in this moduli space. So even if T is a Calabi-Yau threefold it is unlikely to be rigid.

The condition for this not to be possible is that the generic fibre has Picard number 19. If that holds, there is no obvious way to deform T.

So we might hope that T will be rigid if and only if the generic fibre has Picard rank 19. This isn't true, but it's a surprisingly good start.

Calabi-Yau total spaces

If the lattice is too small, the total space will be rational, and if it is too large the space will be more complicated than a Calabi-Yau.

There is something we can try if the total space is rational. Namely, let $d : \mathbb{P}^1 \to \mathbb{P}^1$ be a double cover, and consider a base change of $T \to \mathbb{P}^1$ by d.

If d is chosen randomly we can now deform it, so we don't expect a rigid total space. To avoid this, we choose d so that the double cover is maximally singular (i.e., for all nearby double covers, the singular locus is smaller).

If the double cover is still rational, we can keep doing this.

Prior work in this direction

A paper of Doran-Harder-Novosel'tsev-Thomson classifies the n for which there is a Calabi-Yau threefold whose general fibre is a K3 surface with Picard lattice $E_8 + E_8 + \langle -2n \rangle + U$.

It is a short list; the largest is 23.

They find (somewhat) explicit models for these Calabi-Yau threefolds, from which one could certainly find the modular forms.

I was discouraged by this result for a while, but it turns out that in fact there are a lot of examples where the Picard lattice is not of this form.

Table of Contents

Range of the construction

This construction reproduces many of the known modular forms and finds a fair number of new ones as well. I haven't been systematic about keeping track of the old ones that we rediscover.

In some cases, only nonrigid realizations were previously available, but I find a rigid one.

When we don't take a cover of a rational family, the level of the modular form is usually equal to the discriminant of the lattice (i.e., half the determinant of the Gram matrix). Sometimes this is off by a small square.

In more detail

In addition to a lot of realizations of modular forms already known to arise in rigid Calabi-Yau threefolds, we find, up to level 70:

- 6 apparently rigid realizations of modular forms of weight 4 for which no realization was known before;
- 1 (level 49) nonrigid realization where none was known before;
- 13 apparently rigid realizations where only a nonrigid one was known before.

Many of the last of these arise from a large computer search conducted by Meyer, so this is the first genuine construction in those cases.

There are a few more examples in larger (highly composite) levels, but I need to look at these more systematically.

Again I emphasize that none of my examples has been proved.

One example, concretely

Let $q_1 = x(x+y)(y+z)z, q_2 = (x-y-z-w)(x+y-z+w)yw$.

The total space of the family $tq_1 + uq_2$ in $\mathbb{P}^1 \times \mathbb{P}^3$ is rational.

The double cover given by $t^2 = (-64/27\, q_1 + q_2)(q_2)$ appears to be a rigid Calabi-Yau threefold with the point count formula

$$p^3 + p^2 + (-1 - (-1/p) - (2/p) - (3/p))p + 1 - (2/p)a_p.$$

We could twist by 2 to have just a_p in the constant term.

The formula on the previous page

An empirical formula for the number of points that involves only Artin symbols, powers of p, and the coefficients of a form of weight 4 is strong evidence for rigidity.

I expect that constructing a resolution of singularities will produce a formula of the form $p^3 + f(p)(p^2 + p) + 1 - a_p$, where $f(p)$ involves only Artin symbols. This indicates that $h^{1,0} = h^{2,0} = h^{2,1} = 0$.

The symbols don't have to be quadratic characters; we sometimes have formulas depending on the number of solutions mod p to a polynomial of degree $3, 4, 6$.

It's all right if a form of weight 3 appears in the formula, but if we have p times a form of weight 2 that means a nonrigid realization.

Table of Contents

Things I should still do

The main weakness of this method is that it's hard to prove that the examples really are rigid Calabi-Yau threefolds. I will probably never want to do this for all of them, but I should try

harder at least for two or three.

Also, although many new forms of weight 4 are realized this way, there are still some annoying gaps, including all three forms of level 26. There are still some lattices I haven't investigated fully, because the total space appears to be rational but it is too hard for me to trivialize it explicitly.

One could also try to find nice 1-parameter families of Calabi-Yau threefolds. Families of rank less than 19 can give Calabi-Yau varieties of higher dimension, which might realize modular forms of higher weight.

End of talk

Thank you for your attention.

Are there any questions?