Extending the support of 1- and 2-level densities for cusp form L-functions under square-root cancellation hypotheses.

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Our Object of Interest

Definition

In this talk, the modular forms we consider are *holomorphic cuspidal newforms* $H_k = H_k^\star(1) = H_k^+(1)$ of weight k and level $N = 1$.

We can produce an analytic continuation of the **Modular Form L-function**

$$L(s,f) := \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}$$

where $\lambda_f(n) = a_f(n) n^{-\frac{k-1}{2}}$. Deligne: $\lambda_f(p) \in [-2,2]$.

Density

Definition (1-level density)

The **1-level density** of $L(s, f)$ is

$$D(f;\phi) \ := \ \sum_{j=-\infty}^{\infty} \phi\left(\frac{\log c_f}{2\pi}\gamma_f^{(j)}\right),$$

where $\frac{1}{2} + i\gamma_f^{(j)}$ are zeros of $L(s, f)$ and c_f is the conductor of $L(s, f)$.

We weight contributions of zeros using test functions $\phi : \mathbb{R}^n \to [0, \infty)$ which are even, Schwartz, and $\widehat{\phi}$ having compact support.

Test Functions

Below are two examples:

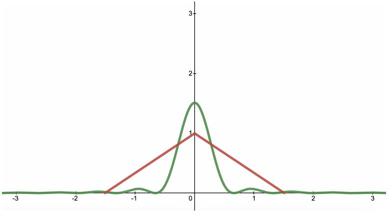

Figure: $\widehat{\phi}$ has smaller support $\Longrightarrow$ *less precise* information about low lying zeros

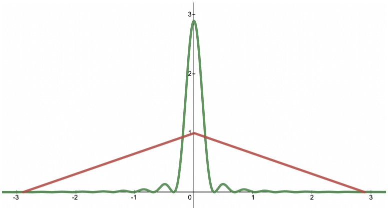

Figure: $\widehat{\phi}$ has larger support $\Longrightarrow$ *more precise* information about low lying zeros

Bounds on Vanishing

By increasing the support of $\hat{\phi}$, we derive better bounds on the percentage of forms (split by sign) which vanish with order at least r.

Density Conjecture (Katz–Sarnak)

If $\mathscr{F}$ is a family of L-functions with symmetry type $\mathcal{G}$, then the average of the density $D(f;\phi)$ over forms $f \in \mathscr{F}$ approaches $\langle \phi, W_{\mathcal{G}} \rangle$ as $|\mathscr{F}| \to \infty$.

Result from ILS

Theorem (ILS)

For $\text{supp}(\widehat{\phi}) \subset (-2,2)$, we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k\leqslant K} \frac{4\pi^2}{k-1} \sum_{f\in \mathrm{H}_k^\star(1)} \omega_f \mathrm{D}(f;\phi) \;=\; \int_{-\infty}^{\infty} \phi(x) W_O(x)\,dx$$

where

$$\omega_f \;:=\; \mathrm{L}^{-1}\left(1, \text{sym}^2(f)\right)$$

are harmonic weights and

$$W_O(x) \;:=\; 1 + \frac{1}{2}\delta_0(x)$$

is a weighting function arising from Random Matrix Theory, where $\mathcal{G} = O$ denotes the orthogonal symmetry type.

Hypothesis S

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For any $x \geqslant 1$, $c \geqslant 1$, and a with $(a, c) = 1$,

$$\sum_{p \leqslant x, p \equiv a(c)} e(2\sqrt{p}/c) \ \ll_\varepsilon \ x^{\frac{1}{2}+\varepsilon}.$$

where $e(z) = e^{2\pi i z}$.

Theorem (ILS)

Assuming Hypothesis S, $\mathrm{supp}(\widehat{\phi}) \subset (-22/9, 22/9)$, we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k \leqslant K} \frac{4\pi^2}{k-1} \sum_{f \in \mathrm{H}_k^\star(1)} \omega_f \mathrm{D}(f;\phi) \ = \int_{-\infty}^{\infty} \phi(x) W_O(x) dx$$

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where $e(z) = e^{2\pi i z}$.

Theorem (ILS)

Assuming Hypothesis S, $\text{supp}(\widehat{\phi}) \subset (-5/2, 5/2)$, we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k \leqslant K} \frac{4\pi^2}{k-1} \sum_{f \in H_k^\star(1)} \omega_f D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W_O(x) dx$$

2-level Density

The **2-level density** is

$$\begin{aligned} D(f;\phi_1,\phi_2) &:= \sum_{\substack{j_1,j_2 \\ j_1\neq\pm j_2}} \phi_1\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_1)}\right)\phi_2\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_2)}\right) \\ &= D(f;\phi_1)D(f;\phi_2) - 2D(f;\phi_1\phi_2). \end{aligned}$$

RMT predicts that

RMT

$$\begin{aligned} \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)} D(f;\phi_1,\phi_2) \approx& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_2(y)W_{O,2}(x,y)\,dxdy \\ &-2\int_{-\infty}^{\infty}\phi_1(t)\phi_2(t)W_O(t)\,dt. \end{aligned}$$

2-level density

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$$-2\int_{-\infty}^{\infty}\phi_1(t)\phi_2(t)W_O(t)\,dt.$$

As done by Hughes and Miller, the above is true when

$$\text{supp}(\widehat{\phi_1})\times\text{supp}(\widehat{\phi_2}) \subset (-\sigma_1,\sigma_1)\times(-\sigma_2,\sigma_2)$$

and $\sigma_1+\sigma_2 \leq 2$.

A New Hypothesis

Consider the following analogue of *Hypothesis S* for the 2-level density.

Hypothesis T

$$\sum_{\substack{p_1 \le x_1 \\ p_1 \equiv a_1(c)}} \sum_{\substack{p_2 \le x_2 \\ p_2 \equiv a_2(c)}} e\left(\frac{2\sqrt{p_1 p_2}}{c}\right) \ll_\varepsilon c^A (x_1 x_2)^{\alpha+\varepsilon}$$

Theorem (Miller,MM–)

Assuming Hypothesis T, we can extend the support of $(\widehat{\phi_1}, \widehat{\phi_2})$ in the 2-level density asymptotic formula to

$$\sigma_1 + \sigma_2 \le 2 + \frac{6 - 8\alpha}{3 + 2A + 4\alpha}.$$

Proof Sketch

We work in full generality, letting the level $n \in \mathbb{N}$.

Expanding the explicit formula for the residue calculus of the global versus local factors with smoothing function

$$\phi(\vec{x}) = \phi_1(x_1)\cdots\phi_n(x_n)$$

We arrive at "lower level" errors plus an *n*-**level error**

$$\sum_{c=1}^{\infty} c^n \int_2^{\infty}\cdots\int_2^{\infty} \mathcal{E}(\vec{x})\, \psi_c(\vec{x})\, d\vec{x}.$$

Proof Sketch

The error

$$\sum_{c=1}^{\infty} c^n \int_2^{\infty} \cdots \int_2^{\infty} \mathcal{E}(\vec{x})\, \psi_c(\vec{x})\, d\vec{x}$$

consists of functions

$$\mathcal{E}(\vec{x}) \;:=\; \max_{\substack{a_1,\dots,a_n \\ \text{distinct}}} \sum_{\substack{p_1 \le x_1 \\ p_1 \equiv a_1(c)}} \cdots \sum_{\substack{p_1 \le x_1 \\ p_n \equiv a_n(c)}} e\left(\frac{2\sqrt{p_1 \cdots p_n}}{c}\right)$$

and

$$\psi_c(\vec{x}) = (x_1 \cdots x_n)^{-3/4}\, \widehat{\phi}\left(\frac{\vec{x}}{2\log K}\right)\, \hbar\left(\frac{cK^2}{8\pi\sqrt{x_1 \cdots x_n}}\right).$$

Proof Sketch

We apply the n-**level square-root cancellation hypothesis**

$$|\mathcal{E}(\vec{x})| \ll_\varepsilon c^A (x_1 \cdots x_n)^{\alpha+\varepsilon}$$

This is the analogue of Hypothesis S and T.

We integrate by parts:

$$K^{O(\varepsilon)} \sum_{c=1}^{\infty} c^{n+A} \int_2^{P_1} \cdots \int_2^{P_n} (x_1 \cdots x_n)^{\alpha - 7/4} \, \hbar \left(\frac{cK^2}{8\pi \sqrt{x_1 \cdots x_n}} \right) d\vec{x}.$$

Proof Sketch

Because $\hbar\left(\frac{cK^2}{\sqrt{P_1\cdots P_n}}\right)$ is rapidly decaying, we focus on

$$(\star) \quad c \ll \sqrt{P_1 \cdots P_n} K^{-2+\varepsilon}.$$

We substitute $u = x_1 \cdots x_n$, obtaining

$$\sum_{(\star)} c^{n+A} \int_2^{P_1} \cdots \int_2^{P_{n-1}} (x_1 \cdots x_{n-1})^{-1} \int_{c^2 K^{4-2\varepsilon}}^{P_n/x_1\cdots x_{n-1}} u^{\alpha-7/4}\, du.$$

Simplifying,

$$\sum_{(\star)} c^{n+A} K^{O(\varepsilon)} \int_{c^2 K^{4-2\varepsilon}}^{\infty} u^{\alpha-7/4}\, du.$$

Conjectures

We arrive at the following estimate for the *n*-level error.

$$(P_1 \cdots P_n)^{n/2-1/4+A/2+\alpha} K^{-2n-2-2A+O(\varepsilon)}$$

- There is a trivial "zero-level error" of $P_1 \cdots P_n K^{-4}$.
- Since $P_j \ll K^{2\sigma_j}$ where supp $\left(\widehat{\phi}_j\right) \subset (-\sigma_j, \sigma_j)$, we have

$$\sigma \ := \ \sigma_1 + \cdots + \sigma_n \ \leq \ 2 + \frac{6-8\alpha}{2n-1+2A+4\alpha}$$

- The **average support** is $\frac{2}{n} + \theta(\frac{1}{n^2})$.

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$$\sigma \ := \ \sigma_1 + \cdots + \sigma_n \ \leq \ 2 + \frac{6-8\alpha}{2n-1+2A+4\alpha}$$

- The **average support** is $\frac{2}{n} + \theta(\frac{1}{n^2})$ vs. $\frac{2}{n}$ (Cohen et al.)

Results

Assume the strongest hypothesis plausible:

- $\alpha = \frac{1}{2}$ and $A = 0$ for $n = 1$.

Theorem (1-level Extended Support)

For $n = 1$, the 0- and 1-level errors are the *only* errors. Therefore

$$\sigma \leq \min\left\{\frac{5}{2}, \frac{8}{3}\right\} = \frac{5}{2}.$$

This is greater than the claim of 22/9 found in ILS.

Results

Assume the strongest hypothesis plausible:

- $\alpha = \frac{1}{2}$ and $A = 0$ for $n = 1, 2$.

Theorem (2-level Extended Support)

For $n = 2$, we handle one-level error cross terms using Hypothesis S, which gives $\sigma \leq \frac{8}{3}$. For the 2-level error we have calculated $\sigma \leq \frac{12}{5}$. Therefore

$$\sigma \leq \min\left\{\frac{5}{2}, \frac{8}{3}, \frac{12}{5}\right\} = \frac{12}{5}.$$

Conjecture in n-level density

Assume the strongest hypothesis plausible:

- $\alpha = \frac{1}{2}$ and $A = 0$ for all $n \in \mathbb{N}$.

Conjecture (n-level Extended Support)

We conjecture that

$$\sigma \leq 2 + \frac{2}{2n+1}$$

where $\phi(\vec{x}) = \phi_1(x_1) \cdots \phi_n(x_n)$ is an n-dimensional test function.

This gives us an **average support** of $\overline{\sigma} := \frac{2}{n} + \frac{1}{n^2} + O(\frac{1}{n^3})$.

A Dose of Skepticism

It should be noted that similar square-root cancellation for closely related exponential sums is *false*.

For example, there is a closely related explicit formula for

$$A(s) = \sum_{n=1}^{\infty} \lambda(n)\Lambda(n)n^{-s}$$

This produces a related sum $S_q(X) \ll_q X^{3/4+\varepsilon}$, and the power $\alpha = 3/4$ cannot be improved.

References

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