An Invariant Property of Mahler Measure

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(joint work with Prof. Matilde Lalín)

Québec-Maine Number Theory Conference
October 16th, 2022

The definition

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It is the average value of $\log |P|$ over the unit n-torus.

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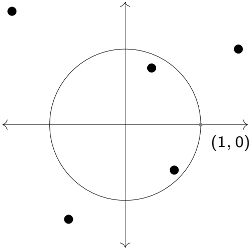

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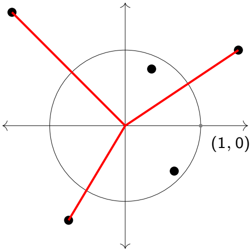

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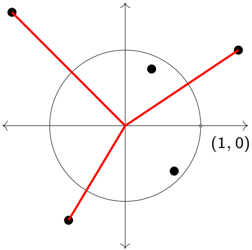

This also means that polynomials with integer coefficients have Mahler measure greater than or equal to zero.

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- Lehmer's Question (1933, still open):

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- The Mahler measure of $P(x)$ is related to heights. For an algebraic integer α with logarithmic Weil height $h(\alpha)$,

$$\mathfrak{m}(f_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]h(\alpha).$$

Not just Number theory, it's everywhere!

The Mahler measure makes an appearance in the following areas

- Knot theory
- Hyperbolic Geometry
- Arithmetic Dynamics
- Height functions

More variables, more problems

In general, calculating the Mahler measure of multi-variable polynomials is much more difficult than the univariate case. However, there are more intriguing results concerning such polynomials that suggest that something deeper is in play.

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We have the Boyd-Lawton formula for any rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$:

$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} \mathfrak{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathfrak{m}(P(x_1, x_2, \ldots, x_n)),$$

where the k_i's vary independently.

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- $$\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

Examples

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- $$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3) = -\frac{112}{5}\zeta'(-2)$$

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- $$\mathfrak{m}\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

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Rogers and Zudilin, 2010:

- $$\mathfrak{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24a3},2) = 4L'(E_{24a3},0)$$

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  For example Boyd conducted many numerical experiments on polynomials of the type

$$A(x) + B(x)y + C(x)z,$$

where A, B and C are products of cyclotomic polynomials.

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We will present a change of variables, which when applied to any polynomial, preserves its Mahler measure

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has all roots outside the unit disc

The Main Result

Theorem (Lalín & N., 2022++)

Let $P(x, y_1, \ldots, y_n)$ *be a polynomial over* $\mathbb{C}$ *in the variables* $x, y_1, \ldots, y_n$*. Let* $g(x) \in \mathbb{C}[x]$ *be such that all the roots have absolute value greater than or equal to one, let* k *be an integer such that* $k > \deg(g)$ *and let* $f(x) = \lambda x^k \overline{g}(x^{-1})$*, where* λ *is a complex number with absolute value one. We denote by* $\widetilde{P}$ *the rational function obtained by replacing* x *by* $f(x)/g(x)$ *in* P*. Then*

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- For eg., with $P = x + 1 + (x-1)(y+z)$, we get

$$\mathfrak{m}(f + g + (f-g)(y+z)) = \frac{28}{5\pi^2}\zeta(3) + \mathfrak{m}(g).$$

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$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

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- Using this theorem, we can obtain the Mahler measure of polynomials with much more complicated geometry

Further questions

- Trying to understand what the $\frac{f}{g}$ transformation means geometrically and how it preserves the L-value.
- Are there any other such transformations that do not change the Mahler measure.
- If the Mahler measure of two polynomials is the same, does that mean they must differ by such a transformation?

HAPPY

TH

BIRTHDAY!