Aspects of dynamical Mahler measure

Annie Carter, Matilde Lalín*, Michelle Manes, Alison Miller, Lucia Mocz

Université de Montréal
matilde.lalin@umontreal.ca
http://www.dms.umontreal.ca/~mlalin

Conférence de théorie des nombres Québec-Maine
in honor of Andrew Granville
Université Laval
October 15th, 2022

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Mahler measure of multivariable polynomials

$P \in \mathbb{C}(x_1, \dots, x_n)^\times$, the (logarithmic) Mahler measure is :

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(z_1, \dots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}. \end{aligned}$$

where $\mathbb{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1\}$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Mahler measure of multivariable polynomials

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) Mahler measure is :

$$\begin{aligned}\mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})| d\theta_1 \ldots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log|P(z_1, \ldots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}.\end{aligned}$$

where $\mathbb{T}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1\}$.

Jensen's formula gives

$$\mathrm{m}(P) = \log|a| + \sum_{|\alpha_i|>1} \log|\alpha_i| \quad \text{if} \quad P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) := \exp(\mathrm{m}(P)).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Mahler measure is ubiquitous!

- Heights
- Distribution of values
- Volumes in hyperbolic space
- Special values of L-functions

Mahler measure is ubiquitous!

- Heights
- Distribution of values
- Volumes in hyperbolic space
- Special values of L-functions

Lehmer's question (1933)
Given $\varepsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < \mathrm{m}(P) < \varepsilon$?

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

$$f : X/K \to X/K \qquad\qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{composition } n \text{ times}} .$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

$$f : X/K \to X/K \qquad\qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{composition } n \text{ times}}.$$

Given $\alpha \in X(K)$, the (forward) f-orbit of α is

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \geq 0\}.$$

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

$$f : X/K \to X/K \qquad\qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{composition } n \text{ times}}.$$

Given $\alpha \in X(K)$, the (forward) f-orbit of α is

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \geq 0\}.$$

- α is periodic if $f^n(\alpha) = \alpha$ for some $n > 0$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

$$f : X/K \to X/K \qquad\qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{composition } n \text{ times}}.$$

Given $\alpha \in X(K)$, the (forward) f-orbit of α is

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \geq 0\}.$$

- α is periodic if $f^n(\alpha) = \alpha$ for some $n > 0$.
- α is preperiodic if $f^n(\alpha) = f^m(\alpha)$ for some $n > m \geq 0$.

Université de Montréal

Centre de Recherches Mathématiques

Arithmetic dynamics

Arithmetic dynamics studies the iterations of morphisms in varieties and the orbits of points that are interesting from the number theoretic point of view.

$$f : X/K \to X/K \qquad\qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{composition } n \text{ times}}.$$

Given $\alpha \in X(K)$, the (forward) f-orbit of α is

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \geq 0\}.$$

- α is periodic if $f^n(\alpha) = \alpha$ for some $n > 0$.
- α is preperiodic if $f^n(\alpha) = f^m(\alpha)$ for some $n > m \geq 0$.
- α is wandering otherwise.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

- The Fatou set of f is the maximal open set for which the family $\{f^n\}_{n\geq 1}$ is equicontinuous.

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

- The Fatou set of f is the maximal open set for which the family $\{f^n\}_{n\geq 1}$ is equicontinuous.
- The Julia set of f is the complement of the Fatou set. We write J_f.

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

- The Fatou set of f is the maximal open set for which the family $\{f^n\}_{n\geq 1}$ is equicontinuous.
- The Julia set of f is the complement of the Fatou set. We write J_f.
- The filled Julia set of a polynomial f is the set of all z such that $f^n(z) \not\to \infty$. We write K_f. We have $J_f = \partial K_f$.

Université de Montréal

Centre de recherches mathématiques

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

- The Fatou set of f is the maximal open set for which the family $\{f^n\}_{n\geq 1}$ is equicontinuous.
- The Julia set of f is the complement of the Fatou set. We write J_f.
- The filled Julia set of a polynomial f is the set of all z such that $f^n(z) \not\to \infty$. We write K_f. We have $J_f = \partial K_f$.

Informally, points in J_f tend to wander away from one another as f is iterated, so f behaves chaotically on its Julia set.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

The Julia set

Let $f : X \to X$ be a morphism in the metric space X. (For simplicity, $X = \mathbb{P}^N$.)

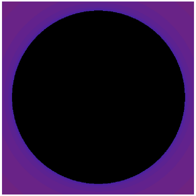
- The Fatou set of f is the maximal open set for which the family $\{f^n\}_{n\geq 1}$ is equicontinuous.
- The Julia set of f is the complement of the Fatou set. We write J_f.
- The filled Julia set of a polynomial f is the set of all z such that $f^n(z) \not\to \infty$. We write K_f. We have $J_f = \partial K_f$.

Informally, points in J_f tend to wander away from one another as f is iterated, so f behaves chaotically on its Julia set.

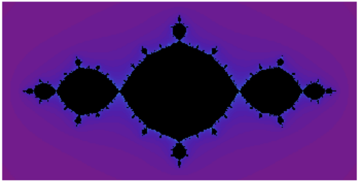
Really informally, the Julia set is where the action is. Dynamically speaking.

Université de Montréal
CENTRE DE RECHERCHES MATHÉMATIQUES

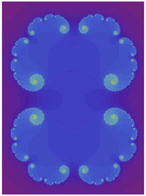
Pretty pictures!



(a) Filled Julia set for $f(z) = z^2$



(b) Filled Julia set for $f(z) = z^2 - 1$



(c) (Filled) Julia set for $f(z) = z^2 + 0.3$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Equilibrium measures

Brolin (1965), Lyubich (1983), Freire-Lopes-Mañé (1983)
Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ polynomial of degree $d \geq 2$. There is a unique Borel probability measure $\mu = \mu_f$ in $\mathbb{P}^1$ such that

- μ is invariant under (the push-forward by) f:

$$f_*\mu = \mu, \qquad f_*(\mu(B)) = \mu(f^{-1}(B))$$

- $\mathrm{Supp}(\mu) = J_f$;
- μ has maximal energy

$$I(\mu) := \int_{J_f}\int_{J_f} \log|z-w| d\mu(z) d\mu(w),$$

among all Borel probability measures on J_f.

Université de Montréal

Centre de Recherches Mathématiques

Equilibrium measures

Brolin (1965), Lyubich (1983), Freire-Lopes-Mañé (1983)
Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ polynomial of degree $d \geq 2$. There is a unique Borel probability measure $\mu = \mu_f$ in $\mathbb{P}^1$ such that

- μ is invariant under (the push-forward by) f:

$$f_*\mu = \mu, \qquad f_*(\mu(B)) = \mu(f^{-1}(B))$$

- $\mathrm{Supp}(\mu) = J_f$;
- μ has maximal energy

$$I(\mu) := \int_{J_f}\int_{J_f} \log|z-w| d\mu(z) d\mu(w),$$

among all Borel probability measures on J_f.

μ is the equilibrium measure of f or of J_f.

Université de Montréal
CENTRE DE RECHERCHES MATHÉMATIQUES

A dynamical Mahler measure

If $f \in \mathbb{Z}[z]$ is monic, the f-dynamical Mahler mesure of $P \in \mathbb{C}(x_1, \dots x_n)^\times$ is given by

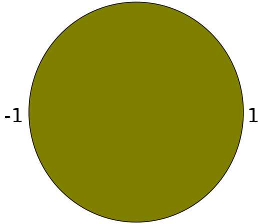
$$\mathrm{m}_f(P) = \int \cdots \int \log |P(z_1, \cdots, z_n)| d\mu_f(z_1) \cdots d\mu_f(z_n).$$

The integral converges and $\mathrm{m}_f(P) \geq 0$ when $P \in \mathbb{Z}[x_1, \dots, x_n]$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

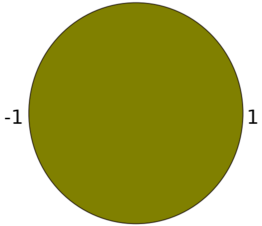
The circle



$$f(z) = z^d$$

- $|\alpha| > 1 \Rightarrow |\alpha^{d^n}| \to \infty$.
- $|\alpha| < 1 \Rightarrow |\alpha^{d^n}| \to 0$.
- $|\alpha| = 1 \Rightarrow |\alpha^{d^n}| = 1$.

The circle



$f(z) = z^d$

- $|\alpha| > 1 \Rightarrow |\alpha^{d^n}| \to \infty$.
- $|\alpha| < 1 \Rightarrow |\alpha^{d^n}| \to 0$.
- $|\alpha| = 1 \Rightarrow |\alpha^{d^n}| = 1$.

$J_f = \{|z| = 1\}$,

$$\mu_f = \frac{\chi_{\mathbb{S}^1} dz}{2\pi i z}, \qquad \mathrm{m}_f(P) = \mathrm{m}(P).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

$$T_d \circ T_e(z) = T_{de}(z) = T_e \circ T_d(z).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

$$T_d \circ T_e(z) = T_{de}(z) = T_e \circ T_d(z) \Rightarrow T_d^n(z) = T_{d^n}(z).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

$$T_d \circ T_e(z) = T_{de}(z) = T_e \circ T_d(z) \Rightarrow T_d^n(z) = T_{d^n}(z).$$

- $\alpha \in [-2, 2] \Rightarrow T_d(\alpha) \in [-2, 2]$.
- $\alpha \notin [-2, 2] \Rightarrow |T_d^n(\alpha)| = |T_{d^n}(\alpha)| \to \infty$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

$$T_d \circ T_e(z) = T_{de}(z) = T_e \circ T_d(z) \Rightarrow T_d^n(z) = T_{d^n}(z).$$

- $\alpha \in [-2, 2] \Rightarrow T_d(\alpha) \in [-2, 2]$.
- $\alpha \notin [-2, 2] \Rightarrow |T_d^n(\alpha)| = |T_{d^n}(\alpha)| \to \infty$.

$J_f = [-2, 2]$,

$$\mu_f = \frac{\chi_{[-2,2]} dx}{\pi\sqrt{4 - x^2}},$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Chebyshev polynomials

$f(z) = T_d(z)$, for $n \geq 2$, where T_d is the d-Chebyshev polynomial

$$T_d(z + z^{-1}) = z^d + z^{-d}$$

$$T_d(z) = \begin{cases} 2 & d = 0, \\ z & d = 1, \\ zT_{d-1}(z) - T_{d-2}(z) & d \geq 2. \end{cases}$$

$$T_d \circ T_e(z) = T_{de}(z) = T_e \circ T_d(z) \Rightarrow T_d^n(z) = T_{d^n}(z).$$

- $\alpha \in [-2, 2] \Rightarrow T_d(\alpha) \in [-2, 2]$.
- $\alpha \notin [-2, 2] \Rightarrow |T_d^n(\alpha)| = |T_{d^n}(\alpha)| \to \infty$.

$J_f = [-2, 2]$,

$$\mu_f = \frac{\chi_{[-2,2]}dx}{\pi\sqrt{4 - x^2}}, \qquad \mathrm{m}_f(P) = \mathrm{m}\left(P \circ \left(z + z^{-1}\right)\right).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

A two-variable case

$f \in \mathbb{Z}[z]$ monic of degree $d \geq 2$.

$$\mathrm{m}_f(x - y) = \int\int \log|z_1 - z_2| d\mu_f(z_1) d\mu_f(z_2) = 0.$$

The energy $I(\mu_f)$ of the equilibrium measure is 0 when f is monic.

Dynamical Kronecker's Lemma

Kronecker (1857)

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathrm{m}(P) = 0 \Longleftrightarrow P(x) = x^n \prod \Phi_i(x)$$

where the Φ_i are cyclotomic polynomials.

Dynamical Kronecker's Lemma

Kronecker (1857)

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathrm{m}(P) = 0 \Longleftrightarrow P(x) = x^n \prod \Phi_i(x)$$

where the Φ_i are cyclotomic polynomials.

Dynamical version CLMMM (2022) $f \in \mathbb{Z}[z]$ monic of degree $d \geq 2$.
$P(x) = a\prod_j (x - \alpha_j) \in \mathbb{Z}[x]$.

$$\mathrm{m}_f(P) = 0 \Longleftrightarrow |a| = 1 \text{ and } \alpha_j \text{ preperiodic}$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Dynamical Boyd–Lawton Theorem

Boyd (1981), Lawton (1983)

For $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathrm{m}(P(x_1, \ldots, x_n))$$

con $k_2, \ldots, k_n \to \infty$ independently.

Dynamical Boyd–Lawton Theorem

Boyd (1981), Lawton (1983)

For $P \in \mathbb{C}(x_1, \dots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \dots, x^{k_n})) = \mathrm{m}(P(x_1, \dots, x_n))$$

con $k_2, \dots, k_n \to \infty$ independently.

(Weak) Dynamical version CLMMM (2022) Let $f \in \mathbb{Z}[z]$ monic of degree $d \geq 2$. and $P \in \mathbb{C}[x, y]$,

$$\limsup_{n \to \infty} \mathrm{m}_f(P(x, f^n(x))) \leq \mathrm{m}_f(P(x, y)).$$

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Lehmer (1933)

Is there $\varepsilon > 0$ *such that if* $P \in \mathbb{Z}[x]$,

$$\mathrm{m}(P) > 0 \Rightarrow \mathrm{m}(P) > \varepsilon?$$

Dynamical Lehmer's Conjecture

Lehmer (1933)

Is there $\varepsilon > 0$ *such that if* $P \in \mathbb{Z}[x]$,

$$\mathrm{m}(P) > 0 \Rightarrow \mathrm{m}(P) > \varepsilon?$$

Dynamical version *Is there* $\varepsilon = \varepsilon_f > 0$ *such that if* $P \in \mathbb{Z}[x]$,

$$\mathrm{m}_f(P) > 0 \Rightarrow \mathrm{m}_f(P) > \varepsilon?$$

Multivariable Kronecker's Lemma

Everest–Ward (1999)

If $P \in \mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$ is primitive (coprime coefficients),

$$\mathrm{m}(P) = 0 \Longleftrightarrow P \text{ is the product of a monomial and } \Phi_{n_i} \text{ evaluated in monomials.}$$

Two-variable Dynamical Kronecker's Lemma

Theorem (CLMMM (2022, 2022+))

Let $f \in \mathbb{Z}[z]$ monic of degree $d \geq 2$, not conjugate to z^d nor $\pm T_d(z)$.

Assume either the Dynamical Lehmer's Conjecture or that $\mathrm{PrePer}(f) \subset J_f$.

Let $P \in \mathbb{Z}[x, y]$ irreducible in $\mathbb{Z}[x, y]$ (with both variables)

$$\mathrm{m}_f(P) = 0 \Leftrightarrow P \text{ divides in } \mathbb{C}[x, y] \text{ a product of } \tilde{f}^n(x) - L(\tilde{f}^m(y)),$$

$L \in \mathbb{C}[z]$ is linear and commutes with an iterate of f and
$\tilde{f} \in \mathbb{C}[z]$ is not linear, commutes with an iterate of f and has minimal degree.

The proof uses a result of unlikely intersections due to Ghioca, Nguyen & Ye (2019).

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that

$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that
$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$
- Show that
$$\{\mathrm{ord}_\alpha P(x, f^n(x))\}_n$$
is bounded for every $\alpha \in \mathrm{PrePer}(f)$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that

$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$

- Show that

$$\{\mathrm{ord}_\alpha P(x, f^n(x))\}_n$$

  is bounded for every $\alpha \in \mathrm{PrePer}(f)$.

- $\{\alpha : P(\alpha, f^n(\alpha)) = 0 \text{ for some } n\}$ is infinite.

Université de Montréal

Centre de Recherches Mathématiques

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that

$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$

- Show that

$$\{\mathrm{ord}_\alpha P(x, f^n(x))\}_n$$

  is bounded for every $\alpha \in \mathrm{PrePer}(f)$.

- $\{\alpha : P(\alpha, f^n(\alpha)) = 0 \text{ for some } n\}$ is infinite.
  This gives infinitely many $(\alpha, f^n(\alpha))$ preperiodic under $f \times f$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that

$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$

- Show that

$$\{\mathrm{ord}_\alpha P(x, f^n(x))\}_n$$

  is bounded for every $\alpha \in \mathrm{PrePer}(f)$.

- $\{\alpha : P(\alpha, f^n(\alpha)) = 0 \text{ for some } n\}$ is infinite.
  This gives infinitely many $(\alpha, f^n(\alpha))$ preperiodic under $f \times f$.
- If $\mathrm{PrePer}(f) \subset J_f$, then $\mathrm{m}_f(P(\alpha, y)) = 0$ for any $\alpha \in J_f$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

Ideas in the proof

Assume $\mathrm{m}_f(P(x,y)) = 0$.

- Use Weak Dynamical Boyd–Lawton and Dynamical Lehmer's question to obtain that

$$\mathrm{m}_f(P(x, f^n(x))) = 0 \text{ for } n \gg 0.$$

- Show that

$$\{\mathrm{ord}_\alpha P(x, f^n(x))\}_n$$

  is bounded for every $\alpha \in \mathrm{PrePer}(f)$.

- $\{\alpha : P(\alpha, f^n(\alpha)) = 0 \text{ for some } n\}$ is infinite.
  This gives infinitely many $(\alpha, f^n(\alpha))$ preperiodic under $f \times f$.
- If $\mathrm{PrePer}(f) \subset J_f$, then $\mathrm{m}_f(P(\alpha, y)) = 0$ for any $\alpha \in J_f$.
  This gives infinitely many $(\alpha, f^n(\alpha))$ preperiodic under $f \times f$.

Université de Montréal

CENTRE DE RECHERCHES MATHÉMATIQUES

A key result

Ghioca, Nguyen & Ye (2019)
Let $f \in \mathbb{C}[z]$ of degree $d \geq 2$, not conjugate to z^d nor $\pm T_d(z)$.

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$\Phi(x, y) = (f(x), f(y)).$$

Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ an irreducible curve over $\mathbb{C}$ which projects dominantly onto both coordinates.

Université de Montréal
CENTRE DE RECHERCHES MATHÉMATIQUES

A key result

Ghioca, Nguyen & Ye (2019)
Let $f \in \mathbb{C}[z]$ of degree $d \geq 2$, not conjugate to z^d nor $\pm T_d(z)$.

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$\Phi(x, y) = (f(x), f(y)).$$

Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ an irreducible curve over $\mathbb{C}$ which projects dominantly onto both coordinates.

Then C contains infinitely many preperiodic points under the action of Φ if and only if C is an irreducible component of the locus of an equation of the form

$$\tilde{f}^n(x) = L(\tilde{f}^m(y)),$$

where $L, \tilde{f} \in \mathbb{C}[z]$ as before.

Looking ahead

- More general f?

Looking ahead

- More general f?
- More concrete examples and formulas.

Looking ahead

- More general f?
- More concrete examples and formulas.
- Dynamical Boyd–Lawton (full strength, more variables).

Looking ahead

- More general f?
- More concrete examples and formulas.
- Dynamical Boyd–Lawton (full strength, more variables).
- Remove conditions for Dynamical Kronecker, also more than two variables.

Looking ahead

- More general f?
- More concrete examples and formulas.
- Dynamical Boyd–Lawton (full strength, more variables).
- Remove conditions for Dynamical Kronecker, also more than two variables.
- Dynamical Lehmer's Question!

Happy birthday Andrew!!!

Joyeux anniversaire Andrew !!!

¡¡¡Feliz cumpleaños Andrew!!!