Binary Theta Series and CM Modular Forms

1. Introduction

Let $r_n(q) = \{(x,y) \in \mathbb{Z}^2 : q(x,y) = n\}$ denote the number of representations of $n \in \mathbb{Z}$ by the positive definite binary quadratic form

$$q(x,y) = ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{Z},\ a > 0.$$

Fermat, Euler, Lagrange, Gauss: When is $r_n(q) > 0$?

Dirichlet(1839), Weber(1882): If $\gcd(a,b,c) = 1$, i.e. if q is primitive, then $\exists_\infty$ primes $p : r_p(q) > 0$. — Study:

$$Z_q(s) = \sum_{n \geq 1} r_n(q) n^{-s}.$$

Following Jacobi, Hermite, Kronecker, Weber, consider the closely related binary theta series

$$\vartheta_q(z) = \sum_{x,y \in \mathbb{Z}} e^{2\pi i q(x,y) z} = \sum_{n \geq 0} r_n(q) e^{2\pi i n z}.$$

Theorem 0 (a) Weber(1893): Let $D = \Delta(q) = b^2 - 4ac$ denote the discriminant of q and $\psi_D = \left(\frac{D}{\cdot}\right)$. Then

$$\vartheta_q\left(\frac{az+b}{cz+d}\right) = \psi_D(d)(cz+d)\vartheta_q(z), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|).$$

(b) Hecke(1926), Schoeneberg(1939): ϑ_q is holomorphic at the cusps, so $\vartheta_q \in M_1(|D|, \psi_D)$.

Fix: a discriminant $D < 0$. Thus:

$$D = f_D^2 d_K, \quad \text{where } K = \mathbb{Q}(\sqrt{D}), d_K = \mathrm{disc}(K), \text{ and } f_D \geq 1$$

is some integer. Let

$$\Theta_D := \langle \vartheta_q : q \in Q_D \rangle_{\mathbb{C}} \quad \text{and} \quad \Theta(D) := \langle \vartheta_q : q \in Q(D) \rangle_{\mathbb{C}}$$

be the $\mathbb{C}$-subspaces generated by the theta-series, where

$$\begin{aligned} Q(D) &= \{q = (a,b,c) \in \mathbb{Z}^2 : a > 0, \Delta(q) = D/t^2\}, \\ Q_D &= \{q = (a,b,c) \in Q(D) : \gcd(a,b,c) = 1, \Delta(q) = D\}. \end{aligned}$$

Thus

$$\Theta_D \subset \Theta(D) \subset M_1(|D|, \psi_D).$$

Questions: 1) How large are the spaces Θ_D and $\Theta(D)$? What is the dimension of the subspaces of cusp forms, i.e. of

$$\Theta_D^S = \Theta_D \cap S_1(|D|, \psi_D) \text{ and } \Theta(D)^S = \Theta(D) \cap S_1(|D|, \psi_D)?$$

Hecke (1926): $\Theta(D) \neq M_1(|D|, \psi_D)$, for many D's.

2) How can a binary theta series ϑ_q be expressed in terms of the (extended) Atkin-Lehner basis of $M_1(|D|, \psi_D)$?

3) Is there an intrinsic characterization of these spaces?

2. Some Observations

1) The group $\mathrm{GL}_2(\mathbb{Z})$ acts on the sets Q_D and $Q(D)$, and

$$\vartheta_{q'} = \vartheta_q, \quad \text{for all } q' \in q\,\mathrm{GL}_2(\mathbb{Z}).$$

By using the Dirichlet/Weber result, one can show that the set $\{\vartheta_q : q \in Q_D/\,\mathrm{GL}_2(\mathbb{Z})\}$ is a basis of Θ_D. In particular,

$$\dim \Theta_D \;=\; \overline{h}_D := |Q_D/\,\mathrm{GL}_2(D)|$$

2) By Gauss's theory of composition of forms, the set

$$\mathrm{Cl}(D) = Q_D/\,\mathrm{SL}_2(\mathbb{Z})$$

has the structure of an abelian group. If $h_D := |\,\mathrm{Cl}(D)|$, then

$$\overline{h}_D = \frac{1}{2}(g_D + h_D), \quad \text{where } g_D = [\mathrm{Cl}(D) : \mathrm{Cl}(D)^2]$$

denotes the number of genera of forms of discriminant D.

3) For a character $\chi \in \mathrm{Cl}(D)^*$ on $\mathrm{Cl}(D)$, put

$$\vartheta_\chi(z) \;:=\; \frac{1}{w_D} \sum_{q \in \mathrm{Cl}(D)} \chi(q)\vartheta_q(z) \;= \sum_{n \geq 0} a_n(\chi) e^{2\pi i n z} \in\; \Theta_D,$$

where $w_D = 2$ for $D < -4$ and $w_{-3} = 6, m_{-4} = 4$.

It is immediate that $\{\vartheta_\chi\}_{\chi \in \mathrm{Cl}(D)^*}$ generates Θ_D and hence by 1) forms a basis of Θ_D (subject to the identification $\vartheta_{\overline{\chi}} = \vartheta_\chi$).

Note: It turns out (cf. Theorem 1) that the coefficients $a_n(\chi)$ are multiplicative in n, and that hence ϑ_χ is a Hecke eigenfunction w.r.t. to the Hecke algebra $\mathbb{T}(D)$ generated by the Hecke operators T_p with $(p, D) = 1$.

4) The L-function associated to the form ϑ_χ is

$$L(s,\chi) \;=\; L(s,\vartheta_\chi) \;=\; \sum_{n\geq 1} a_n(\chi) n^{-s}.$$

This function is frequently found in the literature (e.g., in Lang, *Elliptic Functions*, 1^{st} ed.), and was recently studied in detail by Z.-H. Sun and K. S. Williams (2006).

5) If D is a fundamental discriminant, i.e. if $D = d_K$, then it is well-known that each ϑ_χ is a primitive form (newform) and hence in this case the ϑ_χ's are part of the canonical Atkin-Lehner basis of $M_1(|D|, \psi_D)$.

However, in the general case this is no longer true for every $\chi \in \mathrm{Cl}(D)^*$ because some of the characters $\chi \in \mathrm{Cl}(D)^*$ are not primitive, i.e. they are lifts

$$\chi = \chi' \circ \pi \quad \text{of characters} \quad \chi' \in \mathrm{Cl}(D')^*$$

of some "lower level" $D'|D$ (where $\frac{D}{D'} = t^2 > 1$) via the canonical map

$$\pi = \pi_{D,D'} : \mathrm{Cl}(D) \to \mathrm{Cl}(D').$$

3. Main Results

Theorem 1: The space Θ_D is a $\mathbb{T}(D)$-submodule of $M_1(|D|, \psi_D)$ of multiplicity one, and has a canonical basis $\{\vartheta_\chi\}$ consisting of normalized $\mathbb{T}(D)$-eigenforms. Furthermore, ϑ_χ is a cusp form if and only if χ is not a quadratic character.

Theorem 2: We have $\Theta_D = \Theta_D^E \oplus \Theta_D^S$, where $\Theta_D^E = \Theta_D \cap E_1(|D|, \psi_D)$ denotes the Eisenstein space part and $\Theta_D^S = \Theta_D \cap S_1(|D|, \psi)$ denotes the cusp space part of Θ_D, and

(1) $$\dim \Theta_D^E = g_D \quad \text{and} \quad \dim \Theta_D^S = \tfrac{1}{2}(h_D - g_D).$$

Remark: Thus $\Theta_D^S = 0 \Leftrightarrow h_D = g_D \overset{def}{\Leftrightarrow} D$ is an idoneal discriminant. (This implies a result of Kitaoka (1971).)

Theorem 3: Let $\chi \in \mathrm{Cl}(D)^*$, where $D = f_D^2 d_K$.

(a) $\exists!$ divisor $f_\chi | f_D$ and a unique primitive character $\chi_{pr} \in \mathrm{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$, such that $\chi = \chi_{pr} \circ \bar{\pi}_{D,D_\chi}$.

(b) The form $\vartheta_{\chi_{pr}} \in \Theta_{D_\chi}$ is a primitive form (newform) of level $|D_\chi|$. Moreover, there exist constants $c_n(\chi) \in \mathbb{R}$ such that

(2) $$\vartheta_\chi(z) \ = \sum_{n | \bar{f}_\chi^2} c_n(\chi) \vartheta_{\chi_{pr}}(nz),$$

where $\bar{f}_\chi = f_D / f_\chi$. Furthermore, the function $n \mapsto c_n(\chi)$ is multiplicative and has generating function

(3) $$C(s, \chi) \ := \sum_{n | \bar{f}_\chi^2} c_n(\chi) n^{-s} \ = \ L(s, \vartheta_\chi)/L(s, \vartheta_{\chi_{pr}}).$$

Remark: While $L(s, \vartheta_{\chi_{pr}})$ is a classical Hecke L-function associated to a Hecke character and hence is well-understood, the L-function $L(s, \vartheta_\chi)$ is more complicated and is, in fact, unknown in general.

Thus, (3) does not help in determining the constants $c_n(\chi)$. However, $C(s, \chi)$ can be computed directly by using facts about ideals in quadratic orders.

As a consequence, we thus obtain an explicit expression for the L-function $L(s, \chi) = L(s, \vartheta_\chi)$:

Corollary: If $\chi \in \mathrm{Cl}(D)^*$, then $L(s, \chi)$ has the Euler product

$$L(s,\chi) = \prod_p L_p(s,\chi) \tag{4}$$

where for $p \nmid \bar{f}_\chi$ the p-Euler factor $L_p(s, \chi)$ is given by

$$\begin{aligned} L_p(s,\chi) &= \left(1 - a_p(\chi)p^{-s} + \psi_D(p)p^{-2s}\right)^{-1} \\ &= \left(1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}\right)^{-1}, \end{aligned}$$

whereas for $p \mid \bar{f}_\chi$ (and $p^{\bar{e}_p} || \bar{f}_\chi$), it is given by

$$L_p(s,\chi) = \frac{1 - p^{(1-2s)\bar{e}_p}}{1 - p^{1-2s}} + \frac{\left(1 - \frac{1}{p}\psi_{D_\chi}(p)\right) p^{(1-2s)\bar{e}_p}}{1 - a_p(\chi_{pr})p^{-s} + \psi_{D_\chi}(p)p^{-2s}}.$$

Remark: This generalizes the work of Sun and Williams (2006) (for $D < 0$), who obtained a formula for the p-Euler factors of $L(s, \chi)$ in the case that the class group $\mathrm{Cl}(D)$ is cyclic.

Definition: Let $f \in M_k(N, \psi)$ be a $\mathbb{T}(N)$-eigenfunction with eigencharacter $\lambda_f : \mathbb{T}(D) \to \mathbb{C}$. We say that f has CM (complex multiplication) by a Dirichlet character θ if

$$\lambda_f(T_p)\theta(p) = \lambda_f(T_p), \quad \text{for all } p \nmid N\mathrm{cond}(\theta),$$

or, equivalently, if

$$\lambda_f(T_p) = 0 \quad \text{for all } p \nmid N\mathrm{cond}(\theta) \text{ with } \theta(p) \neq 1.$$

We let $M_k^{CM}(N, \psi; \theta)$ denote the space generated by all $T(N)$-eigenfunctions $f \in M_k(N, \psi)$ which have CM by θ.

Theorem 4: For every discriminant $D < 0$ we have that

$$(5) \qquad \Theta(D) \;=\; M_1^{CM}(|D|, \psi_D) \;:=\; M_1^{CM}(|D|, \psi_D; \psi_D).$$

Corollary:

$$(6) \qquad \dim \Theta(D) = \dim M_1^{CM}(|D|, \psi_D) = \sum_{f | f_D} 2^{\omega(f)} \overline{h}_{D/f^2},$$

where $\omega(f)$ denotes the number of distinct prime divisors of f. Moreover, the dimensions of the Eisenstein part and of the cuspidal part of $M_1^{CM}(|D|, \psi_D)$ are given by

$$\dim E_1^{CM}(|D|, \psi_D) \;=\; \sum_{f | f_D} 2^{\omega(f)} g_{D/f^2},$$

$$\dim S_1^{CM}(|D|, \psi_D) \;=\; \sum_{f | f_D} 2^{\omega(f)} (f_{D/f^2} - g_{D/f^2}).$$

Remark: There is no (known) formula for $\dim M_1(|D|, \psi_D)$.

4. Ingredients

1) Dedekind's Isomorphism:

$$\lambda_D : \mathrm{Cl}(D) \xrightarrow{\sim} \mathrm{Pic}(\mathfrak{O}_D),$$

where $\mathfrak{O}_D = \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2} \subset \mathfrak{O}_K$ is the order of discriminant D (and/or of conductor f_D in K).

2) A classification of the invertible ideals of $\mathfrak{O}_D$:

$\Rightarrow$ the multiplicativity of $a_n(\chi)$,
the value of $c_n(\chi)$ for $n|D$, etc.

3) A study of the conductor of $\chi \in \mathrm{Cl}(D)^*$: via the isomorphism

$$I_K(f_D\mathfrak{O}_K)/P_{K,\mathbb{Z}}(f_D) \xrightarrow{\sim} \mathrm{Pic}(\mathfrak{O}_D),$$

one can identify each $\chi \in \mathrm{Cl}(D)^*$ with a Hecke character $\tilde{\chi}$ on the group $I_K(f_D\mathfrak{O}_K)$ of fractional ideals prime to the ideal $f_D\mathfrak{O}_K$. A key fact is:

$$\chi \text{ is primitive on } \mathrm{Cl}(D) \Leftrightarrow \tilde{\chi} \text{ is primitive mod } f_D\mathfrak{O}_K.$$

4) Genus theory (Gauss/Kronecker/Weber): this identifies quadratic characters $\chi \in \mathrm{Cl}(D)^*$ with certain Dirichlet characters.

5) Extended Atkin-Lehner theory: this describes:

1) the characters $\lambda \in \mathbb{T}(N)^* = \mathrm{Hom}(\mathbb{T}(N), \mathbb{C})$ of the Hecke algebra $\mathbb{T}(N) \subset \mathrm{End}(M_k(N,\psi))$ in terms of primitive eigenfunctions (newforms);

2) the structure of the $\mathbb{T}(N)$-eigenspace associated to λ:

$$M_k(N,\psi)[\lambda] = \{f \in M_k(N,\psi) : f|_kT_n = \lambda(T_n)f, \forall (n,N) = 1\}$$

For Theorem 4, we also need:

6) (a) The Deligne/Serre theory of Galois representations

$$\rho_f : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

attached to $\mathbb{T}(N)$-eigenfunctions $f \in M_1(N, \psi)$.

(b) A characterization of characters of ring class fields via (strongly) dihedral Galois representations of $G_{\mathbb{Q}}$ (= reinterpretation of a result of Bruckner (1966)).

(c) A characterization of CM forms via their associated Galois representations ($\rightarrow$ Theorem 5 below).

5. Galois representations

Deligne/Serre (1974): If $f \in M_1(N,\psi)$ is a normalized $\mathbb{T}(N)$-eigenfunction, then $\exists!$ Galois representation

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

such that for all primes $p \nmid N$

$$\begin{aligned} \mathrm{tr}(\rho_f(Fr_p)) &= \lambda_f(T_p) = a_p(f), \\ \det(\rho_f(Fr_p)) &= \psi(p). \end{aligned}$$

Furthermore, ρ_f is irreducible $\Leftrightarrow$ f is a cusp form.

Definition: An Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is called strongly dihedral if $\mathrm{Im}(\rho) \simeq D_n$ is a dihedral group ($n \geq 3$).

Moreover, ρ is said to be of dihedral type if $\mathrm{Im}(\rho)/Z(\mathrm{Im}(\rho)) \simeq D_n$ is a dihedral group ($n \geq 2$).

Theorem 5: Let $f \in S_1(N,\psi)$ be a newform.

(a) f has CM by some character $\theta \Leftrightarrow \rho_f$ is of dihedral type.

(b) f has CM by $\psi \Leftrightarrow \rho_f$ is strongly dihedral.

Theorem 6: Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be Galois representation.

(a) (Hecke, Weil, Deligne/Serre) If ρ is of dihedral type and is odd, then $\rho = \rho_f$ for some $f \in S_1(N,\psi)$.

(b) (Bruckner, 1966) ρ is strongly dihedral if and only if the field $\mathrm{Fix}(\mathrm{Ker}(\rho))$ is contained in some ring class field.

Remark: Theorems 3, 5, 6 $\Rightarrow$ Theorem 4.