The distribution of some arithmetic sequences in arithmetic progressions to large moduli

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To count primes, we usually define

 $\pi(x) := \#\{p \le x\}.$

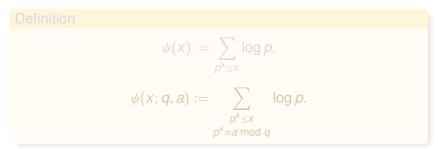
For technical reasons, we add the weight log *p* at each prime *p*.

Definition $\psi(x) := \sum_{p^k \le x} \log p,$ $\psi(x; q, a) := \sum_{\substack{p^k \le x \\ p^k \equiv a \mod q}} \log p.$

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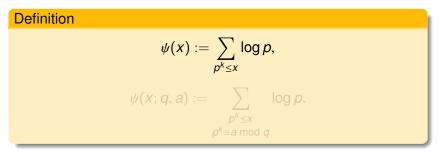
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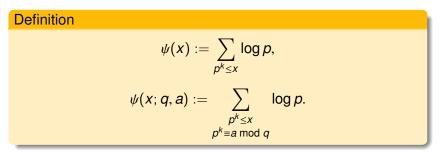
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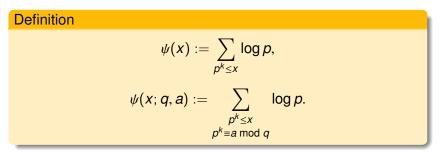
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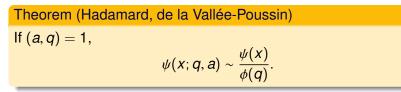
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The prime number theorem in arithmetic progressions



This is for fixed values of *a* and *q*. What if we want to look at higher moduli ?

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Theorem (Hadamard, de la Vallée-Poussin)

If (a,q)=1, $\psi(x;q,a)\sim rac{\psi(x)}{\phi(q)}.$

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Higher moduli

Theorem (Siegel, Walfisz) If (a, q) = 1, then for $q \le (\log x)^B$, $\left| \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right| \le C \frac{x}{(\log x)^A}.$

Theorem

Assume GRH. If (a, q) = 1, then

$$|\psi(x; q, a) - \frac{\psi(x)}{\phi(q)}| \le C\sqrt{x}(\log x)^2.$$

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On average, we know much more.

Theorem (Bombieri, Vinogradov) For $Q \le x^{\frac{1}{2}-\epsilon}$, $\sum_{q \le Q} \max_{a:(a,q)=1} \left| \psi(x;q,a) - \frac{\psi(x)}{\phi(q)} \right| \le C \frac{x}{(\log x)^A}.$

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Instead of looking at the mean deviation, look at the mean itself.

Theorem (F.)

Let $a \neq 0$, and $M = M(x) \leq (\log x)^B$. We have that

$$\frac{1}{\frac{x}{M}\frac{\phi(a)}{a}}\sum_{\substack{q\leq\frac{x}{M}\\(q,a)=1}} \left(\psi(x;q,a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)}\right) = \mu(a,M) + O\left(\frac{1}{M^{\frac{205}{538}-\epsilon}}\right)$$

where

 $\mu(a, M) := \begin{cases} -\frac{1}{2} \log M - C_5 & \text{ if } a = \pm 1, \\ -\frac{1}{2} \log p & \text{ if } a = \pm p^e, \\ 0 & \text{ if } a \text{ has } \ge 2 \text{ distinct prime factors.} \end{cases}$

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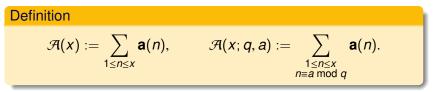
Fix a sequence $\mathcal{R} = {\mathbf{a}(n)}_{n \ge 1}$ a sequence of non-negative real numbers.

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In each of the sequences we will consider, there exists $g_a(q)$ such that

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For $\alpha, \beta, \gamma \in \mathbb{Z}$ coprime, let

$$Q(x, y) := \alpha x^2 + \beta x y + \gamma y^2$$

be a positive definite binary quadratic form. The discriminant: $d := \beta^2 - 4\alpha\gamma$.

$$\mathbf{a}(n) := \#\{(x, y) \in \mathbb{Z}^2_{\geq 0} : Q(x, y) = n\}.$$

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What happens in arithmetic progressions ?

$$\mathcal{A}(x;q,a) \sim \frac{\rho_a(q)}{q} \mathcal{A}(x).$$
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This asymptotic actually holds in great uniformity.

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Assume that $d \equiv 1, 5, 9, 12, 13 \mod 16$. Fix *a* such that (a, 2d) = 1. We have for $M = M(x) \le x^{\lambda}$, where $\lambda < \frac{1}{12}$ that

$$\begin{split} \frac{1}{x/M} \sum_{q \leq \frac{x}{M}} \left(\mathcal{A}(x;q,a) - \mathbf{a}(a) - \frac{\rho_a(q)}{q} \mathcal{A}(x) \right) \\ &= -C_Q \rho_a(4d) r_d(|a|) + O\left(\frac{1}{M^{\frac{1}{3}-\epsilon}}\right), \end{split}$$

where
$$C_Q := \frac{A_Q}{2L(1,\chi_d)} \qquad \left(= \frac{w_d \sqrt{|d|}}{4\pi h_d} A_Q\right),$$

 A_Q = area of { $(x, y) \in \mathbb{R}^2_{\geq 0}$: $Q(x, y) \leq 1$ }, $\chi_d := \left(\frac{4d}{\cdot}\right)$, w_d is the number of units of $\mathbb{Q}(\sqrt{d})$ and h_d is its class number.

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$\mathbf{a}(n) := \Lambda(n)\Lambda(n+2).$

Conjecture (Hardy-Littlewood)

 $\mathcal{A}(x)\sim 2C_2x,$

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Twin primes

The (general) Hardy-Littlewood actually tells something about arithmetic progressions.

For (a, q) = 1, look at

$$B(x) := \sum_{n \le x} \Lambda(qn+a) \Lambda(qn+a+2).$$

The Hardy-Littlewood prediction is that

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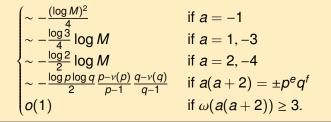
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Under a uniform version of Hardy-Littlewood, the average of $\mathcal{A}(x; q, a) - \mathbf{a}(a) - \frac{\mathcal{A}(x)}{q\gamma(q)}$ for $\frac{x}{2M} < q \leq \frac{x}{M}$ is



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$$\begin{cases} \sim -\frac{(\log M)^2}{4} & \text{if } a = -1 \\ \sim -\frac{\log 3}{4} \log M & \text{if } a = 1, -3 \\ \sim -\frac{\log 2}{2} \log M & \text{if } a = 2, -4 \\ \sim -\frac{\log p \log q}{2} \frac{p - \nu(p)}{p - 1} \frac{q - \nu(q)}{q - 1} & \text{if } a(a + 2) = \pm p^e q^f \\ o(1) & \text{if } \omega(a(a + 2)) \ge 3. \end{cases}$$

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Sums of two squares, without multiplicity

Define

$$\mathbf{a}(n) := egin{cases} 1 & ext{if } n = \Box + \Box, \\ 0 & ext{else.} \end{cases}$$

In this case,

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where, for $p \neq 2$ with $p^f \parallel a$,

$$g_{a}(p^{e}) := \frac{1}{p^{e}} \times \begin{cases} 1 & \text{if } p \equiv 1 \mod 4 \\ 1 & \text{if } p \equiv 3 \mod 4, e \leq f, 2 \mid e \\ \frac{1}{p} & \text{if } p \equiv 3 \mod 4, e \leq f, 2 \nmid e \\ 1 + \frac{1}{p} & \text{if } p \equiv 3 \mod 4, e > f, 2 \mid f \\ 0 & \text{if } p \equiv 3 \mod 4, e > f, 2 \nmid f. \end{cases}$$
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Moreover, $g_a(2) := \frac{1}{2}$ and for $e \ge 2$, $g_a(2^e) := \frac{1+(-1)^{\frac{a-1}{2}}}{2^{e+2}}$

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Theorem (F.))

Fix an integer $a \equiv 1 \mod 4$. We have for $1 \le M(x) \le (\log x)^{\lambda}$, where $\lambda < 1/5$ is a fixed real number, that

$$\frac{1}{x/2M} \sum_{\frac{x}{2M} < q \le \frac{x}{M}} \left(\mathcal{A}(x;q,a) - \mathbf{a}(a) - g_a(q) \mathcal{A}(x) \right) \\ \sim - \left(\frac{\log M}{\log x} \right)^{\frac{1}{2}} \frac{(-4)^{-l_a - 1} (2l_a + 2)!}{(4l_a^2 - 1)(l_a + 1)!\pi} \prod_{\substack{p^f \mid | a: \\ p \equiv 3 \mod 4, \\ fodd}} \frac{\log(p^{\frac{f+1}{2}})}{\log M}, \quad (3)$$

where $l_a := \#\{p^f \mid | a : p \equiv 3 \mod 4, 2 \nmid f\}$ is the number primes dividing *a* to an odd power which are congruent to 3 modulo 4.

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$$\mathcal{A}(x, y) := \sum_{\substack{n \le x \\ p < y}} \mathbf{a}_{y}(n),$$
$$\gamma_{y}(q) := \prod_{\substack{p \mid q \\ p < y}} \left(1 - \frac{1}{p}\right),$$
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Theorem (F.)

Let $M \leq (\log x)^{1-\delta}$. The average of $\mathcal{R}(x, y; q, a) - \mathbf{a}_y(a) - \frac{\mathcal{R}(x, y)}{q\gamma_y(q)}$

for $x/2M < q \le x/M$ is, for $y \le e^{(\log M)^{\frac{1}{2}-\delta}}$ with $y \to \infty$,

 $= \begin{cases} -\frac{1}{2} + o(1) & \text{if } a = \pm 1\\ o(1) & \text{otherwise,} \end{cases}$

and for $(\log x)^{\log \log \log x} \le y \le \sqrt{x}$, it is

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D. Fiorilli Arithmetic sequences in arithmetic progressions

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