New methods for constructing normal numbers

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Given an integer $q \geq 2$, a *q-normal number* is an irrational number whose q-ary expansion is such that any preassigned sequence, of length $k \geq 1$, of base q digits from this expansion, occurs at the expected frequency, namely $1/q^k$.

Equivalently, given a positive irrational number $\eta < 1$ whose expansion is

$$\eta = 0.a_1a_2a_3\cdots = \sum_{j=1}^{\infty} \frac{a_j}{q^j}, \text{ where each } a_i \in \{0, 1, \ldots, q-1\},$$

we say that η is a normal number if the sequence $\{q^m\eta\}$, $m = 1, 2, \ldots$ (here $\{y\}$ stands for the fractional part of y), is uniformly distributed in the interval $[0, 1[$.

Both definitions are equivalent, because the sequence $\{q^m\eta\}$, $m = 1, 2, \ldots$, is uniformly distributed in $[0, 1[$ if and only if for every integer $k \geq 1$ and $b_1 \ldots b_k \in \{0, 1, \ldots, q-1\}^k$, we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{j < N : a_{j+1} \ldots a_{j+k} = b_1 \ldots b_k\} = \frac{1}{q^k}.$$

Interestingly:

- π, e, $\sqrt{2}$, $\log 2$ and $\zeta(3)$ have not yet been proven to be normal numbers.
- In fact, no algebraic irrational number has yet been proved to be normal.
- Émile Borel (1909) showed that almost all numbers are normal (with respect to the Lebesgue measure).

The story line:

- 1909: Borel introduces the concept of a normal number.
- 1917: Sierpinski provides an example of a normal number.
- 1933: Champernowne proves that the number

  $$0.1234567891011121314151617181920 21\ldots,$$

  is normal in base 10.
- 1946: Copeland and Erdős prove that the number

  $$0.23571113171923293137\ldots$$

  is normal in base 10.

  In the same paper, they conjecture that if $f(x)$ is any non constant polynomial whose values at $x = 1, 2, 3, \ldots$ are positive integers, then $0.f(1)f(2)f(3)\ldots$ is a normal number in base 10.
- 1952: Davenport and Erdős prove this conjecture.
- 1997: Nakai and Shiokawa prove the following: Let $f(x)$ be any nonconstant polynomial taking only positive integral values for positive integral arguments, then the number $0.f(2)f(3)f(5)f(7)\ldots f(p)\ldots$ is normal.
- 2003: Crandall and Bailey prove that, if $b > 1$ and $c > 1$ are co-prime integers, then

  $$\sum_{k=1}^{\infty} \frac{1}{b^{c^k}\ c^k} \qquad \text{is } b\text{-normal.}$$
- 2010: Igor Shparlinski asks if the number

  $$0.P(2)P(3)P(4)P(5)P(6)\ldots$$

  is normal. Here $P(n)$ stands for the largest prime factor of n.

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Here, we use the complexity of the multiplicative structure of integers to construct large families of normal numbers.

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Let $q \geq 2$ be a fixed integer.

$\wp =$ the set of all primes.

Let $\wp_0, \wp_1, \ldots, \wp_{q-1}$ be disjoint sets of primes such that

$$\wp = \mathcal{R} \cup \wp_0 \cup \wp_1 \cup \cdots \cup \wp_{q-1}, \tag{1}$$

where $\mathcal{R}$ is a given finite (perhaps empty) set of primes.

We call $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ a *disjoint classification of primes.*

Example of a disjoint classification of primes:

$$\mathcal{R} = \{2\}, \quad \wp_0 = \{p : p \equiv 1 \pmod 4\}, \quad \wp_1 = \{p : p \equiv 3 \pmod 4\}$$

The general idea:

$$n = p_1^{a_1} \cdots p_r^{a_r} \mapsto \ell_1 \ldots \ell_r,$$

where each ℓ_j is such that $p_j \in \wp_{\ell_j}$

For each integer $q \geq 2$, let $A_q := \{0, 1, \ldots, q-1\}$.

Given an integer $t \geq 1$, an expression of the form

$$i_1 i_2 \ldots i_t, \qquad \text{where each } i_j \in A_q$$

is called a *word* of length t.

The symbol Λ will denote the *empty word.*

Now, given a disjoint classification of primes $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$, let the function $H : \wp \to A_q$ be defined by

$$H(p) = \begin{cases} j & \text{if } p \in \wp_j \text{ for some } j \in A_q, \\ \Lambda & \text{if } p \in \mathcal{R} \end{cases}$$

Let A_q^* be the set of finite words over A_q.

Consider the function $T : \mathbb{N} \to A_q^*$ defined by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r),$$

where we omit $H(p_i) = \Lambda$ if $p_i \in \mathcal{R}$.

For convenience, we set $T(1) = \Lambda$.

Given a set of integers S, we let

$$\pi(S) = \#\{p \in \wp \cap S\}.$$

Theorem 1. *Let $q \geq 2$ be an integer and let $\wp = \mathcal{R} \cup \wp_0 \cup \ldots \cup \wp_{q-1}$ be a disjoint classification of primes. Assume that, for a certain constant $c \geq 5$,*

$$\pi([u, u+v] \cap \wp_j) = \frac{1}{q}\pi([u, u+v]) + O\left(\frac{u}{\log^c u}\right)$$

uniformly for $2 \leq v \leq u$, $j = 0, 1, \ldots, q-1$, as $u \to \infty$. Further, let T be defined on $\mathbb{N}$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r),$$

where

$$H(p) = \begin{cases} j & \text{if } p \in \wp_j \text{ for some } j \in A_q, \\ \Lambda & \text{if } p \in \mathcal{R} \end{cases}$$

Then,

$$\xi = 0.T(1)T(2)T(3)T(4)\ldots$$

is a q-normal number.

Example: Let

$$\wp_0 = \{p : p \equiv 1 \pmod 4\}, \quad \wp_1 = \{p : p \equiv 3 \pmod 4\}, \quad \mathcal{R} = \{2\}.$$

Then,

$$\{T(1), T(2), \ldots, T(15)\} = \{\Lambda, \Lambda, 1, \Lambda, 0, 1, 1, \Lambda, 1, 0, 1, 1, 0, 1, 10\}$$

and

$$\xi = 0.T(1)T(2)T(3)T(4)\ldots = 0.101110110110\ldots \qquad \text{is a normal number}$$

Theorem 2. *Given two co-prime positive integers a and D, let $\wp_h := \{p : p \equiv h \pmod D\}$ for $\gcd(h, D) = 1$. Let $h_0, h_1, \ldots, h_{\varphi(D)-1}$ be those positive integers $< D$ which are relatively prime with D. Further let $\mathcal{R} = \{p : p | D\}$ and set*

$$T(p^a) = T(p) = \begin{cases} j & \text{if } p \equiv h_j \pmod D, \\ \Lambda & \text{if } p | D. \end{cases}$$

Let ξ be the real number whose $\varphi(D)$-ary expansion is given by

$$\xi = 0.T(2+a)T(3+a)T(5+a)\ldots T(p+a)\ldots,$$

where $p + a$ is the sequence of shifted primes. Then ξ is a $\varphi(D)$-normal number.

Theorem 3. *Let $k \geq 2$ be a fixed integer and set $E(n) := n(n+1)\cdots(n+k-1)$. Moreover, for each positive integer n, define*

$$e(n) = \prod_{\substack{q^\beta \| E(n) \\ q \leq k-1}} q^\beta.$$

We shall now define the sequence ρ_n on the prime powers q^a of $E(n)$ as follows:

$$\rho_n(q^a) = \rho_n(q) = \begin{cases} \Lambda & \text{if } q | e(n), \\ \ell & \text{if } q | n+\ell,\ gcd(q, e(n)) = 1,\ 0 \leq \ell \leq k-1. \end{cases}$$

If $E(n) = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ where $q_1 < q_2 < \cdots < q_r$ are primes an each $a_i \in \mathbb{N}$, then we set

$$S(E(n)) = \rho_n(q_1)\rho_n(q_2)\ldots\rho_n(q_r).$$

Let ξ be the real number whose k-ary expansion is given by

$$\xi = 0.S(E(1))S(E(2))\ldots S(E(n))\ldots \tag{2}$$

Then, ξ is a k-normal number.

Theorem 4. *Let $p_1 < p_2 < \cdots$ be the sequence of all primes, and let k, E and S be as above. Let ξ be the real number whose k-ary expansion is given by*

$$\xi = 0.S(E(p_1+1))S(E(p_2+1))\ldots$$

Then ξ is a k-normal number.

Theorem 5. *Let $q \geq 2$ be a fixed integer. Given a positive integer*

$$n = p_1^{e_1} \cdots p_{k+1}^{e_{k+1}},$$

let

$$c_j(n) := \left\lfloor \frac{q \log p_j}{\log p_{j+1}} \right\rfloor \in A_q \qquad (j = 1, \ldots, k).$$

Define the arithmetic function H by

$$H(n) = H(p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}) = \begin{cases} c_1(n)\ldots c_k(n) & \text{if } \omega(n) \geq 2, \\ \Lambda & \text{if } \omega(n) \leq 1. \end{cases}$$

Then the number

$$\xi = 0.H(1)H(2)H(3)\ldots$$

is a q-normal number.

Given a positive integer n, write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where each $\varepsilon_i(n) \in A_q$ and $\varepsilon_t(n) \neq 0$. Then write

$$\begin{aligned} \overline{n} &= \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \\ \overline{\overline{n}} &= \varepsilon_t(n)\varepsilon_{t-1}(n)\ldots\varepsilon_0(n) \end{aligned}$$

Theorem 6. *Let $F \in \mathbb{Z}[x]$ be a primitive polynomial with positive leading coefficient and of positive degree. Then the numbers*

$$\xi = 0.\overline{F(P(2))}\,\overline{F(P(3))}\ldots\overline{F(P(p))}\ldots$$

and

$$\xi^* = 0.\overline{\overline{F(P(2))}}\,\overline{\overline{F(P(3))}}\ldots\overline{\overline{F(P(p))}}\ldots$$

are normal.

Theorem 7. *Let F be as in Theorem 6. Then the numbers*

$$\eta = 0.\overline{F(P(2+1))}\,\overline{F(P(3+1))}\ldots\overline{F(P(p+1))}\ldots$$

and

$$\widetilde{\eta} = 0.\overline{\overline{F(P(2+1))}}\,\overline{\overline{F(P(3+1))}}\ldots\overline{\overline{F(P(p+1))}}\ldots$$

are normal.

One of the key result being used is the following:

Theorem A (JMDK & IK, Acta Arith., 1995) *Let* $\mathcal{R}, \wp_0, \wp_1, \dots, \wp_{q-1}$ *be a disjoint classification of primes such that*

$$\pi([u, u+v]|\wp_i) = \delta_i \pi([u, u+v]) + O\left(\frac{u}{(\log u)^{c_1}}\right) \tag{1.1}$$

holds uniformly for $2 \le v \le u$, $i = 0, 1, \dots, q-1$, *where* $c_1 \ge 5$ *is a constant,* $\delta_0, \delta_1, \dots, \delta_{q-1}$ *are positive constants such that* $\sum_{i=0}^{q-1} \delta_i = 1$. *Let* $\lim_{x\to\infty} w_x = +\infty$, $w_x = O(x_3)$, $\sqrt{x} \le Y \le x$ *and* $1 \le k \le c_2 x_2$, *where* c_2 *is an arbitrary constant. Let* $A \le x_2$ *with* $P(A) \le w_x$. *Then,*

$$\#\{n = An_1 \le Y : p(n_1) > w_x,\ \omega(n_1) = k,\ H(n_1) = i_1 \dots i_k\}$$

$$= (1 + o(1))\delta_{i_1} \cdots \delta_{i_k} \frac{Y}{A \log Y} \frac{x_2^{k-1}}{(k-1)!} \varphi_{w_x}\left(\frac{k-1}{x_2}\right) F\left(\frac{k-1}{x_2}\right),$$

where

$$\varphi_w(z) := \prod_{p \le w} \left(1 + \frac{z}{p}\right)^{-1} \quad \text{and} \quad F(z) := \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p}\right)\left(1 - \frac{1}{p}\right)^z.$$

References:

- J.M. De Koninck and I. Kátai, *Construction of normal numbers by classified prime divisors of integers*, to appear in Functiones et Approximatio.
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