Repeated Values of Euler's Function

a talk by Paul Kinlaw on joint work with
Jonathan Bayless

Oct 4 2015

Problem of Erdős: Consider solutions of the equations

$$\varphi(n) = \varphi(n+k), \qquad \sigma(n) = \sigma(n+k)$$

for fixed $k \in \mathbb{N}$, particularly $k = 1$.

Currently Unsolved: Are there are infinitely many solutions?

Schinzel and Sierpiński (1958): The sequence $\varphi(n+1)/\varphi(n)$ is dense in $(0, \infty)$. (Similar result for σ).

Erdős-Mirsky Conjecture (Proved by Heath-Brown, 1984): There infinitely many n for which $d(n+1) = d(n)$. The same has been proved for ω and Ω.

Notation: Let $A = \{n \in \mathbb{N} : \varphi(n) = \varphi(n+1)\}$ and $A(x) = \#\{n \le x : n \in A\}$. Similarly, let $B = \{n \in \mathbb{N} : \sigma(n) = \sigma(n+1)\}$ and let $B(x) = \#\{n \le x : n \in B\}$.

Theorem (Erdős, 1936:) The asymptotic density of n with $\varphi(n) < \varphi(n+1)$ is 1/2, and the same for $\varphi(n) > \varphi(n+1)$. Thus A has asymptotic density 0. The same holds for B.

Theorem (Erdős, Pomerance and Sárközy, 1987): $A(x) < x/\exp(\sqrt[3]{\log x})$ for sufficiently large x, and the same for B. It follows that

$$\sum_{n \in A} \frac{1}{n} < \infty, \qquad \sum_{n \in B} \frac{1}{n} < \infty$$

They conjecture that $A(x) > x^{1-\epsilon}$ for every $\epsilon > 0$ and $x \ge x_0(\epsilon)$, and similarly for B.

Theorem (Bayless, K): We have

$$1.4324884 < \sum_{n \in A} \frac{1}{n} < 441702.$$

Also,

$$0.080958 < \sum_{n \in B} \frac{1}{n} < 610838.$$

We have recently improved the upper bounds to 95153 for A and 607740 for B.

Def: A *multiperfect* number is a number n such that $n|\sigma(n)$. Let M denote the set of multiperfect numbers and $M(x)$ its counting function. It is conjectured that M is infinite.

Theorem (Erdős, 1956): $M(x) \leq x^{0.75+\epsilon}$ for all $\epsilon > 0$ and $x \geq x_0(\epsilon)$.

Theorem (Bayless, K):

$$\sum_{n \in M} \frac{1}{n} = 1.21440760859142719 \ldots.$$

Proposition (Bayless, K): There are infinitely many n such that $\varphi(n) = \varphi(n+k)$ for some $k < n^{0.2962}$. The same holds for σ.

Proof: Baker and Harman proved that

$$|\varphi^{-1}(m)| > m^{0.7039}$$

infinitely often. It is also known if $\varphi(n) = m$ then $n < 3m \log\log m < m^{1+\epsilon}$ for all $\epsilon > 0$ and sufficiently large m. Thus $\varphi^{-1}(m) \subset [1, m^{1+\epsilon}]$ has size greater than $m^{0.7039}$. The result follows by the Pigeon Hole principle.

Proposition (Bayless, K): If $\varphi(n) = \varphi(n+1)$ then $\frac{\varphi(n)}{n} < 0.5$ with precisely six exceptions: $n = 1,\ 3,\ 15,\ 255,\ 65535,\ 4294967295.$

Proposition (Bayless, K): If $\varphi(n) = \varphi(n+1)$ then $\varphi(n)/n < 0.5$ with precisely six exceptions, when n is the square-free product of the first k Fermat primes, $k = 0, \ldots, 5$.

Sketch of Proof: We have

$$\frac{\varphi(n)}{n} = \frac{\varphi(n+1)}{n+1}\left(1 + \frac{1}{n}\right).$$

Thus by the product formula for φ,

$$\prod_{p|n}\left(1 - \frac{1}{p}\right) = \left(1 + \frac{1}{n}\right)\prod_{q|n+1}\left(1 - \frac{1}{q}\right).$$

Consider three cases, n is even, n is odd but $n+1$ is not a power of two, or $n+1$ is a power of two. (The third case yields the 6 exceptions.)

Proposition (Bayless, K): If $\sigma(n) = \sigma(n+1)$ then $\sigma(n)/n > 1.5$.

Sketch of Proof: Use the estimate

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \le \frac{\sigma(n)}{n} < \prod_{p|n}\left(1+\frac{1}{p-1}\right)$$

valid for $n > 1$. Consider three cases: n is odd, n is even but not a power of 2, or n is a power of two (the third case cannot occur).

Explicit Bounds on the sum of reciprocals of n such that $\varphi(n) = \varphi(n+1)$.

Small Range: $n \le 10^{12}$. An exhaustive list of all 5236 solutions in this range was computed by Noe and McCranie (OEIS). We compute to obtain

$$\sum_{\substack{n\in A\\ n\le 10^{12}}} \frac{1}{n} = 1.4324884\ldots$$

Middle Range: $10^{12} < n \le e^{10^6}$.
We need a very large cutoff to get a good upper bound on the large range. We feared the worst:

$$\sum_{10^{12} < n \le e^{10^6}} \frac{1}{n} < 10^6.$$

However Carl Pomerance suggested to us that solutions $n \equiv 1 \pmod 3$ appear to be rare.

Observation: All solutions $1 < n \le 10^{12}$ are congruent to 2 or 3 (mod 6), with relative frequencies very close to 0.5. We cannot find any exceptions $n > 10^{12}$. This led us to the following result.

Lemma (Bayless, K): Suppose $n > 1$ and $\varphi(n) = \varphi(n+1)$.
(1.) If $n \equiv 5 \pmod 6$ then $\omega(n) \ge 33$.
(2.) If $n \equiv 0 \pmod 6$ then $\omega(n+1) \ge 33$.
(3.) If $n \equiv 1 \pmod 6$ and $\gcd(n, 35) = 1$ then $\omega(n) \ge 27$.
(4.) If $n \equiv 4 \pmod 6$ and $\gcd(n+1, 35) = 1$ then $\omega(n+1) \ge 27$.

Explicit Hardy-Ramanujan Inequality

Let $\pi_k(x) = \#\{n \le x : \omega(n) = k\}$.

$\pi_k(x) \sim \dfrac{x(\log\log x)^{k-1}}{(k-1)!\log x}$.

The same asymptotic formula also applies to:

$$\tau_k(x) = \#\{n \le x : \Omega(n) = k\},$$

$$\pi_k^{sf}(x) = \#\{n \le x : \omega(n) = \Omega(n) = k\},$$

$$\upsilon_k(x) = \#\{n \le x : \Omega(n) \le k\}.$$

These are variants of Landau's 1900 theorem, proved for $\tau_k(x)$ by induction on k.

Hardy-Ramanujan Inequality: $C_1, C_2 > 0$.

$$\pi_k(x) \le \frac{C_1 x(\log\log x + C_2)^{k-1}}{(k-1)!\log x}.$$

Theorem (Bayless, Klyve) For $x \ge 10^{12}$,

$$\pi_k^{sf}(x) \le \frac{1.0924x(\log\log x + 0.2622)^{k-1}}{(k-1)!\log x}.$$

Theorem (Bayless, K) For $x \ge 10^{12}$,

$$\pi_k(x) \le \frac{1.0989x(\log\log x + 1.1174)^{k-1}}{(k-1)!\log x}.$$

This bounds the contribution to the middle range from 118 residue classes mod 210.

The Large Range: We follow the proof of Erdős, Pomerance and Sárközy, making all implied constants explicit as we proceed.

Five classes of numbers are dealt with separately, and then the count of remaining numbers is bounded. We can then apply partial summation to bound the contribution to the reciprocal sum.

For instance, their proof uses $P(n) \geq L^2$, where

$$L(x) = \exp\left(\frac{1}{8}(\log x)^{2/3} \log\log x\right)$$

and addresses the count of L^2-*smooth* numbers n (i.e. $P(n) \leq L^2$) up to x separately.

De Bruijn's Ψ function:

$$\Psi(x,y) = \#\{n \le x : P(n) \le y\}.$$

Theorem (de Bruijn): There exists $c > 0$ such that

$$\Psi(x,y) \le x \exp(-cu \log u)$$

where $u = \log x / \log y$.

Lemma (Bayless, K): For $x \ge 10^{21}$ we have $\Psi(x,y) \le 1.033x \exp(-0.7u \log u)$.

Conclusion: Our best bound on the reciprocal sum comes from this improvement (addition of the $\log u$ term above), and by adjusting the cutoffs for two different large ranges, as well as other coefficients.

Some open problems.

Improve the bounds on these reciprocal sums.

Do there exist solutions $n > 1$ of the equation $\varphi(n) = \varphi(n+1)$ that are not 2 or 3 mod 6?

Here are the five assumptions made in the proof of Pomerance, Erdős and Sárközy. Counts of integers violating the assumptions are bounded separately. Here

$$\ell(x) = \exp(\sqrt[3]{\log x})$$

and

$$L(x) = \exp\left(\frac{1}{8}(\log x)^{2/3} \log\log x\right).$$

1. $P(n) \geq L^2$ and $P(n+1) \geq L^2$.

2. If k^a divides n or $n+1$ where $a \geq 2$, then $k^a \leq \ell^3$.

3. $\min\{n/P(n), (n+1)/P(n+1)\} \geq L$.

4. $\min\{P(\varphi(m)), P(\varphi(m'))\} \geq \ell^4$, for $m = n/P(n)$ and $m' = (n+1)/P(n+1)$.

5. $P(n) > P(n+1)$.