# The Algebra and Arithmetic of Vector-Valued Modular Forms on $\Gamma_0(2)$

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Thanks to the conference organizers for creating such a fantastic conference!

# Notation

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$H$ = complex upper half plane.
$\tau \in H$
$q = e^{2\pi i \tau}$
$G$ = finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$
$\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is a $n$-dim. complex representation
$k \in \mathbf{Z}$
If $F : H \to \mathbf{C}^n$ is a function then

$$F|_k \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right](z) := (cz + d)^{-k} F(\frac{az + b}{cz + d})$$

# Definition of a vector-valued modular form (vvmf) on a subgroup

A vector-valued modular form (vvmf) of dimension $n$ and weight $k$ with respect to a complex $n$-dimensional representation $\rho$ of a subgroup $G$ of $\mathrm{SL}_2(\mathbf{Z})$ is:

A holomorphic function $F : H \to \mathbf{C}^n$ such that $F$ has a holomorphic $q$-series expansion at all the cusps of $G$ and

$$F|_k\gamma = \rho(\gamma)F \text{ for all } \gamma \in G.$$

# The transformation equation

Let $F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$.

The transformation equation is:

$$\begin{bmatrix} F_1|_k\gamma \\ F_2|_k\gamma \\ \vdots \\ F_n|_k\gamma \end{bmatrix} = \rho(\gamma) \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}.$$

$M_k(\rho) := \{\text{vector-valued modular forms of weight } k \text{ for } \rho\}$.
$M(\rho) := \bigoplus_{k \in \mathbf{Z}} M_k(\rho)$
$M_k(G) := \{\text{modular forms of weight } k \text{ on } G\}$.
$M(G) := \bigoplus_{k \in \mathbf{Z}} M_k(G)$

The vector spaces $M_k(\rho)$ are finite dimensional. Moreover, $M_k(\rho) = 0$ if $k << 0$.

If $F \in M_k(\rho)$ and if $m \in M_t(G)$ then $mF \in M_{k+t}(\rho)$. In this way, we view $M(\rho)$ as a $\mathbf{Z}$-graded $M(G)$-module.

# Goals

Goal 1: Understand the structure of vector-valued modular forms $M(\rho)$ as a $M(G)$-module.
Approaches to this problem use:

Algebraic geometry (Cameron Franc and Luca Candelori)

Adapt the classical theory of modular forms to vector-valued modular forms (Geoff Mason and Chris Marks)

A Riemann-Hilbert perspective (Terry Gannon).

## Theorem

*(Mason-Marks, Gannon, Candelori-Franc.) Let $\rho$ denote a representation of $SL_2(\mathbf{Z})$ of dimension $n$. Then $M(\rho)$ is a free $M(SL_2(\mathbf{Z})) = \mathbf{C}[E_4, E_6]$-module of rank $n$.*

## Theorem

*(Gottesman, Candelori-Franc) Let $G$ denote a finite index subgroup of $SL_2(\mathbf{Z})$ such that there exist $X, Y \in M(G)$ which are algebraically independent and such that $M(G) = \mathbf{C}[X, Y]$. Let $\rho$ denote a representation of $G$. Then $M(\rho)$ is a free $M(G)$-module of rank $dim\ \rho$.*

Examples: The groups $\mathrm{SL}_2(\mathbf{Z})$, $\Gamma_0(2)$, and $\Gamma(2)$ satisfy the hypothesis of this theorem.

Goal 2: If $M(\rho)$ is a free $M(G)$-module, give explicit formulas for a $M(G)$-basis for $M(\rho)$ in terms of classical functions such as hypergeometric series.

There has been extensive progress made on this problem when $\rho$ is an irreducible representation of $\mathrm{SL}_2(\mathbf{Z})$ of small dimension. My work gives explicit formulas in the case when $\rho$ is an irreducible representation of $\Gamma_0(2)$ of dimension two.

Goal 3: Study scalar-valued modular forms via vector-valued modular forms. This is possible since the component functions of a vector-valued modular form with respect to a representation $\rho$ are modular forms on the subgroup $\ker\rho$..

# Noncongruence subgroups

A subgroup of $\mathrm{SL}_2(\mathbf{Z})$ is *congruence* if membership in the subgroup is determined by congruence conditions on the entries of the matrix.
Atkin and Swinnerton-Dyer noticed examples of modular forms on noncongruence subgroups whose sequence of Fourier coefficients have unbounded denominators. One way to attack and to generalize the unbounded denominator conjecture (i.e. "Modular forms on non-congruence subgroups have unbounded denominators") is via vector-valued modular forms.

# The Unbounded Denominator Conjecture for Vector Valued Modular Forms

## Definition

*Let $\alpha$ denote an algebraic number. The denominator of $\alpha$ is the smallest positive integer N such that N$\alpha$ is an algebraic integer. The number N$\alpha$ is the numerator of $\alpha$.*

## Conjecture

*Let $\rho$ be a representation of a finite index subgroup G of $SL_2(\mathbf{Z})$. If $ker\ \rho$ is noncongruence then for any vector valued modular form X with respect to $\rho$, the sequence of the denominators of the Fourier coefficients of at least one of the component functions of X is unbounded. (Provided the Fourier coefficients are algebraic numbers.)*

# Main Result

## Theorem

*(Gottesman) Let $\rho$ denote an irreducible representation of $\Gamma_0(2)$ of dimension two such that $\rho(T)$ has finite order, $\rho$ is induced from a character of $\Gamma(2)$, and such that a certain mild technical hypothesis on $\rho$ holds. Then for any vector-valued modular form $V$ with respect to $\rho$, the sequence of the denominators of the Fourier series coefficients of each component function of $V$ is unbounded provided these Fourier coefficients are algebraic numbers.*

**Remark**: This approach should allow us to eliminate the assumption that $\rho$ is induced from a character of $\Gamma(2)$.

If $\rho$ is an irreducible representation of $\Gamma_0(2)$ of dimension two, if $\rho(T)$ has finite order and if a mild hypothesis on $\rho$ holds then for each $k \in \mathbf{Z}$, there is a basis of $M_k(\rho)$ consisting of vector-valued modular forms which can be normalized so that the Fourier coefficients of their normalization are elements of a certain quadratic field $\mathbf{Q}(\sqrt{M})$, which is determined by $\rho$.

In this context, normalizing means scaling each of the two component functions by a complex number so that the leading Fourier coefficient of each scaled component function is equal to one.

Equivalent formulation: There is a representation $\rho'$, which is conjugate to $\rho$, such that if $\rho$ is an irreducible representation of $\Gamma_0(2)$ of dimension two, if $\rho(T)$ has finite order and if a mild hypothesis on $\rho$ holds then for each $k \in \mathbf{Z}$, there is a basis of $M_k(\rho')$ consisting of vector-valued modular forms whose Fourier coefficients belong to a certain quadratic field $\mathbf{Q}(\sqrt{M})$, which is determined by $\rho$.

## Theorem

*(Gottesman) Assume that $\rho$ is an irreducible representation of $\Gamma_0(2)$ of dimension two such that $\rho(T)$ has finite order, $\rho$ is induced from a character of $\Gamma(2)$, and a mild hypothesis on $\rho$ holds. Let $p$ denote a sufficiently large prime number such that $M$ is not a quadratic residue mod $p$. Let $X$ denote a vector-valued modular form for $\rho'$ whose component functions have algebraic Fourier coefficients which lie in the quadratic field $\mathbf{Q}(\sqrt{M})$. Then $p$ divides the denominator of at least one Fourier coefficient of the first component function and of the second component function of $X$.*

**Remark:** For each $k \in \mathbf{Z}$, we can always find a basis of vector-valued modular forms for $M_k(\rho')$ whose Fourier coefficients lie in the quadratic field $\mathbf{Q}(\sqrt{M})$.

**Remark**: The density of prime numbers $p$ for which $M$ is not a quadratic residue mod $p$ is one half.

It follows that at least one half of the prime numbers divide the denominator of at least one Fourier coefficient of the first and of the second component functions of $X$. In particular, the sequence of the denominators of the Fourier coefficients of the first and of the second component functions of $X$ are both unbounded.

**Remark:** A similar idea should allow us to drop the assumption that $\rho$ is induced from a character of $\Gamma(2)$

*Method of Proof*:

**Step** 1. Prove that the module $M(\rho)$ of vector valued modular forms for a representation $\rho$ of $\Gamma_0(2)$ is a free graded module over the ring $M(\Gamma_0(2))$, the ring of modular forms on $\Gamma_0(2)$, (Note: Not true for many other subgroups of $\mathrm{SL}_2(\mathbf{Z})$.)

**Step** 2. Use Step 1 and the modular derivative to compute a basis for this module when $\rho$ is irreducible and two-dimensional. Let $k_0$ denote the integer for which $M_{k_0}(\rho) \neq 0$ and $M_k(\rho) = 0$ if $k < k_0$. Let $F$ denote a nonzero element in $M_{k_0}(\rho)$. Then $F$ and $D_{k_0}F = q\frac{d}{q}F - \frac{k_0}{12}E_2F$ form a basis for $M(\rho)$ as a $M(\Gamma_0(2))$-module. In particular, $M_{k_0}(\rho) = \mathbf{C}F$ and so $F$ is unique up to scaling by a complex number.

*Method of Proof*:

**Step** 3. Make this basis even more explicit by solving a modular linear differential equation satisfied by a vector-valued modular form $F$ of minimal weight.

In particular, we then obtain formulas for the Fourier series coefficients involve rising factorials or Pochhammer symbols $(r)_n := r(r+1)\cdots(r+n-1) = \text{product of the first n consecutive numbers starting with } r.$

**Step** 4. Apply the arithmetic of quadratic fields to show that certain sets of prime numbers never divide $(r)_n$ for any $n$ where $r$ is a certain type of element in a quadratic field. Then prove that a vector-valued modular form of least weight whose Fourier coefficients are algebraic numbers has unbounded denominators.

**Step** 5. Use the module structure of vector-valued modular forms and step 4 to show that all vector-valued modular forms whose Fourier coefficients are algebraic numbers have unbounded denominators.

## Back to step 3: Solving the differential equation that $F$ satisfies

$M(\rho)$ is a free $M(\Gamma_0(2))$-module of rank two with basis $F \in M_{k_0}(\rho)$ and $D_{k_0} F \in M_{k_0+2}(\rho)$.
As $D_{k_0+2}(D_{k_0} F) \in M_{k_0+4}(\rho)$:

$$D_{k_0+2}(D_{k_0} F) = (\text{weight two}) D_{k_0} F + (\text{weight four}) F.$$

Let $G(\tau) := -E_2(\tau) + 2E_2(2\tau) \in M_2(\Gamma_0(2))$ where $E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$.

$$M(\Gamma_0(2)) = \mathbf{C}[G, E_4]$$

.

$$0 = D_{k_0+2}(D_{k_0} F) + aGD_{k_0} F + (bG^2 + cE_4) F$$

Let $\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ denote the Dedekind $\eta$-function.
Let $\mathfrak{J} := \frac{3G^2}{E_4 - G^2}$. (a Hauptmodul for $\Gamma_0(2)$)
Let $r$ denote a complex number such that $r(r-1) + (\frac{2-3a}{3})r + (b+4c) = 0$. Let $A$ and $B$ denote the roots of the quadratic polynomial $x^2 - x(2r + \frac{1-6a}{6}) + \frac{r-6c}{2}$.
Then a basis of solutions to the differential equation $0 = D_{k_0+2}(D_{k_0}F) + aGD_{k_0}F + (bG^2 + cE_4)F$ is:

$$\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-A}\ {}_2F_1(A, \frac{1}{2}+A, 1+A-B; \mathfrak{J}(\tau)^{-1}),$$

$$\eta^{2k_0}(\tau)(\mathfrak{J}(\tau)-1)^r \mathfrak{J}(\tau)^{-B}\ {}_2F_1(B, \frac{1}{2}+B, 1+B-A; \mathfrak{J}(\tau)^{-1})$$

where ${}_2F_1(\alpha, \beta, \gamma; z) = 1 + \sum_{n\geq 1} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$,
$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$

$$0 = D_{k_0+2}(D_{k_0}F) + aGD_{k_0}F + (bG^2 + cE_4)F.$$

The fact that $D_{k_0}(\eta^{2k_0}) = 0$ implies that

$$0 = D_2\left(D_0\left(\frac{F}{\eta^{2k_0}}\right)\right) + aGD_0\left(\frac{F}{\eta^{2k_0}}\right) + (bG^2 + cE_4)\left(\frac{F}{\eta^{2k_0}}\right).$$

Let $\mathfrak{J}(\tau) := \frac{3G(\tau)^2}{E_4(\tau)-G(\tau)^2}$. The function $\mathfrak{J}$ is a modular function and it induces a complex-analytic isomorphism from $\Gamma_0(2)\backslash(\mathfrak{H}\bigcup\mathbb{P}^1(\mathbf{Q}))$ to $\mathbb{P}^1(\mathbf{C})$. Therefore $\mathfrak{J}$ is locally injective (away from an elliptic point) and we may write $\frac{F}{\eta^{2k_0}} = H \circ \mathfrak{J}$.

$$\frac{F}{\eta^{2k_0}} = H \circ \mathfrak{J}.$$

The differential equation

$$0 = D_2\left(D_0\left(\frac{F}{\eta^{2k_0}}\right)\right) + aGD_0\left(\frac{F}{\eta^{2k_0}}\right) + (bG^2 + cE_4)\left(\frac{F}{\eta^{2k_0}}\right)$$

becomes (let $Y = \mathfrak{J}(\tau)$) a Fuchsian differential equation on the Riemann sphere with regular singularities at $0, 1, \infty$:

$$H^{''}(Y) + \frac{7Y - 6aY - 3}{6Y(Y-1)}H^{'}(Y) + \frac{(b+c)Y + 3c}{Y(Y-1)^2}H(Y) = 0.$$

A basis for the solutions to this ODE can be found using the Gaussian hypergeometric function ${}_2F_1$. We then obtain explicit formulas for $F$.

# Arithmetic of Quadratic Fields

Here is the lemma we use in Step 4:

## Lemma

*Let $M$ denote a square-free integer. Let $p$ denote an odd prime number for which $M$ is not a quadratic residue mod $p$. Let $X \in \mathbf{Q}(\sqrt{M})$ such that $X \notin \mathbf{Q}$. Let $Z$ denote the smallest positive integer such that $ZX$ is an algebraic integer and let $Y := ZX$. (We think of $Z$ as the denominator and $Y$ as the numerator of $X$.) Let $y$ and $z$ denote the integers for which $Y = \frac{x+y\sqrt{M}}{2}$. If $p \nmid y$ then $p$ does not divide the numerator of any element in the set $\{(X)_t : t \geq 1\}$. (i.e. $p$ does not divide $(X)_t$ in the ring of algebraic integers.)*

This lemma allows us to show that all sufficiently large primes $p$ for which $M$ is not a quadratic residue mod $p$ divide the denominator of least one Fourier coefficient of the vector-valued modular form $F$ of minimal weight provided $\rho$ satisfies certain conditions, including the condition that $\rho$ is induced from a character of $\Gamma(2)$.

Thank you!!