Introduction to Green functions
and theta lifting identities

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What is a Green function?

In simple words a Green function \( G \) is the solution to an equation which looks like:

\[
DG = S
\]

where

1) \( S \) is a Dirac distribution
2) \( D \) is a linear partial differential operator which usually involves a "Laplacian like operator"
3) \( G \) is the unique solution to (*) which satisfies some "boundary conditions"
Let \((M, g)\) be a Riemannian manifold
\(x, y \in M\) be variables
\(\Delta_x = \text{Laplacian in the } x \text{ variable } \Rightarrow L^2(M)\)
\(\Delta_x = \text{positive Laplacian so in } \mathbb{R}^n, \Delta_x = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\)

**Examples of Green functions**

1. Let \(D = \frac{\partial}{\partial t} + \Delta_x\) and \(G = U^M(x, y; t)\) be the unique solution to

\[
\left(\frac{\partial}{\partial t} + \Delta_x\right) U^M(x, y; t) = \delta_y(x) \cdot \delta(t)
\]

\(t > 0, \quad x, y \in M\)

\(U^M(x, y; t)\) is called the heat kernel of \(M\)
\( a \) If \( M \) is compact then the spectrum of \( \Delta_x G L^2(M) \) is discrete; say: \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots \)

Then \( U^M(x, y; t) = \sum_{k=1}^{\infty} u_k(x) \overline{u_k(y)} e^{-\lambda_k t} \) where \( \{u_k(x)\}_{k=1}^{\infty} \) is an orthonormal basis of eigenvectors of \( \Delta_x \).

\( b \) If \( M = \mathbb{R}^n \) then spectrum \( (\Delta_x) = [0, \infty) \) is absolutely continuous and
\[
U^{\mathbb{R}^n}(x, y; t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}}
\]
where \( r = |x-y| \)
In general, if \((M_1, g_1)\) and \((M_2, g_2)\) are Riemannian manifolds we always have

\[
U^{M_1 \times M_2}((x_1, x_2), (y_1, y_2); t) = U^{M_1}(x_1, y_1; t) \cdot U^{M_2}(x_2, y_2; t)
\]

\((x_1, x_2), (y_1, y_2) \in M_1 \times M_2\)
(2) \[ D = \Delta_x - \lambda \text{ where } \lambda \text{ is a spectral parameter} \]

Let \( G = G^M(x, y; \lambda) \) be the unique solution to

\[
(\Delta_x - \lambda) G^M(x, y; \lambda) = \delta_y(x)
\]

\( \rightarrow \) resolvent Green function of \( \Delta_x \) on \( L^2(M) \)

It follows from (t) that \( G^M(x, y; \lambda) \) satisfies the following reproducing identity:

\[
(\Delta_x - \lambda) \int_M G^M(x, y; \lambda) f(y) dy = f(x) \text{ for } f \in L^2(M) \text{ and } \lambda \notin \sigma(\Delta_x)
\]
a) If $M$ is compact then

$$G^M(x,y; \lambda) = \sum_{k=1}^{\infty} \frac{u_k(x)u_k(y)}{\lambda - \lambda_k} \quad \text{where } \lambda \in \mathbb{C}, \ Re(\lambda) < 0$$

where $\{u_k(x)\}_{k=1}^{\infty}$ is an orthonormal basis of eigenvectors of $\Delta_x \subset L^2(M)$.

b) If $M = \mathbb{R}^n$ then

$$G^{\mathbb{R}^n}(x,y; \lambda) = (2\pi)^{-n/2} \frac{n-1}{r} \frac{1}{n-2} \frac{K_{n-1}(r\sqrt{\lambda})}{L_n K_n}$$

where $r = |x-y|$.

In particular, $G^{\mathbb{R}^n}(x,y; \lambda) \sim \begin{cases} \frac{-1}{2\pi} \log r & \text{if } n=2 \\ \frac{1}{n(n-2) \text{vol}(B_n)} \frac{1}{r^{n-2}} & \text{if } n \geq 3 \end{cases}$ as $r \to 0$. 
The resolvent Green function \( G^M(x,y;\lambda) \) and the heat kernel are related in the following way:

\[
G^M(x,y;\lambda) = \int_0^\infty U^M(x,y;t) e^{\lambda t} \, dt \quad \text{Re}(\lambda) << 0
\]

and

\[
U^M(x,y;t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G^M(x,y;\lambda) e^{-\lambda t} \, d\lambda \quad \text{for } t > 0 \quad \text{Re}(\epsilon) << 0
\]
c) Consider the space \((M, g) = (\mathbb{H}, ds^2 = \frac{d\xi^2 + dy^2}{y^2})\) where 

\[ h = \{ x + iy \in \mathbb{C} : y > 0 \} \]  

Poincaré upper half-plane

Let \( G^h_s (z_1, z_2) : = G^h (z_1, z_2 ; s(1 - s)) \)

\( L^2 (h) \) resolvent Green function on \( h \)

\[ \Delta z_1 = -y_1^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) \]

where \( z_1 = x_1 + iy_1 \)

\( z_2 = x_2 + iy_2 \)

\[ \text{spectrum} (\Delta z_1) = [\frac{1}{4}, \infty) \] and is absolutely continuous

We have

\[ G^h_s (z_1, z_2) = \frac{\Gamma(s)^3}{4\pi \Gamma(2s)} \left( 1 - \text{th}^2 \frac{t}{2} \right)^s \_2F_1 (s, s, 2s ; 1 - \text{th}^2 \frac{t}{2}) \]

where \( \rho = \text{dist}_{\text{hyp}} (z_1, z_2) \)

\( \text{th} \frac{t}{2} = \left| \frac{z_1 - z_2}{z_1 - \overline{z}_2} \right| \)
Let \( \Gamma \leq \text{SL}_2(\mathbb{R}) \) be a discrete subgroup and let

\[ Y_\Gamma = \Gamma \setminus \mathbb{H} \quad \text{(hyperbolic surface)} \]

Then

\[
\Gamma^Y_{s} (z, \bar{z}) := \sum_{\gamma \in \Gamma} \mathbb{e}^{\lambda \gamma z} \mathbb{e}^{-\lambda \gamma \bar{z}} \quad \forall \lambda \in \mathbb{C}, \quad \Re(\lambda) > 1
\]

is the \( \Gamma \)-group average of the resolvent Green function on \( \mathbb{H} \).
An example: Let $\Gamma = \text{SL}_2(\mathbb{Z})$ and $\chi = \chi \downarrow \text{SL}_2(\mathbb{Z})$.

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \text{Re}(s) > 1.$$ 

$$E(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma_{\infty}} \chi(|z|)$$

$$\text{Im}(yz)^s = y^s + \frac{\zeta(2s-1) \chi(s) y^{1-s}}{\zeta(2s)} + O(\exp(-cy))$$

$$z = x + iy \in \mathcal{H}, \quad \text{Re}(s) > 1.$$ 

$$\gamma \in \Gamma_{\infty}$$

$$\gamma(z, z) = \frac{1-s}{1-2s} E(z, s) + O\left(\exp\left(-cy\right)\right).$$

$$\text{For\text{\large{}er\text{\small{} series expansion of}} [x_2 \rightarrow G^Y_{\delta}(z, x_2 + iy_2)] \rightarrow \text{Fourier series expansion of}}$$
Resolvent Green function's 
Kronecker limit formula

\[ Y = \frac{h}{SL_2(\mathbb{Z})} \]

\[
\lim_{s \to 1} G_s^{(z_1, z_2)} = \frac{-3}{\pi} \frac{1}{s(1-s)} - \frac{i}{2\pi} \log \left| y_1 y_2 P(z_1, z_2) \right| + O(s-1)
\]

where \[ P(z_1, z_2) = \left( j(z_1) - j(z_2) \right) \Delta(z_1) \Delta(z_2) \]

\begin{align*}
\rightarrow & \quad \text{Selberg prime form of } Y \\
\Longrightarrow & \quad \text{It is not too difficult to deduce from (\star) the } \\
\text{First Kronecker limit Formula for } E(z, s) \text{ as } s \to 1 \]

\text{First Kronecker limit Formula for } E(z, s) \text{ as } s \to 1
The Jacquet-Langlands correspondence

Let $B_1 = M_2(\mathbb{Z})$

Let $B_2 = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ where $i^2 = q \neq 0, j^2 = r \neq 0$ where $r, q$ are positive square-free and coprime integers such that $1 = q x^2 + r y^2$ has no natural solution.

It follows that $B_2$ is an order of an indefinite division algebra where the discriminant of the order is $(4qr)^2$.

Let $\Gamma_1 := \Gamma_0(4qr) \leq B_1^*$ and $\Gamma_2$ = reduced norm 1 elements of $B_2$
Let \( Y_j = \mathcal{H} \) for \( j = 1, 2 \).

\[ \Gamma_j \]

\( Y_1 \) is non-compact while \( Y_2 \) is compact.

The surfaces \( Y_1 \) and \( Y_2 \) are "arithmetically close" from one another since \( \mathcal{B}_1 \otimes \mathbb{Z} \mathcal{O}(\sqrt{7}, v_F) \cong \mathcal{B}_2 \otimes \mathbb{Z} \mathcal{O}(\sqrt{7}, v_F) \).

Key fact: It follows from Jacquet–Langlands correspondence (or the Selberg trace formula) that

\[
\left\{ \text{spectrum } \Delta \subset L^2(Y_2) \right\} \subseteq \left\{ \text{discrete spectrum } \Delta \subset L^2(Y_1) \right\}
\]
Let $\mathcal{D}_j \subset \mathbb{H}$ be a fundamental domain for $\Gamma_j$ ($j = 1, 2$).

It was noticed by John Fogg that such a spectral correspondence "is manifested" through the following "theta lifting identity":

$$
\int_{\mathcal{D}_1} \Theta(\tau, z_0) G_\mathcal{D}_1(\tau, z) \, d\mu(\tau) = \int_{\mathcal{D}_2} \Theta(z, w) G_\mathcal{D}_2(z_0, w) \, d\mu(w)
$$

where $\Theta(\tau, z)$ is a suitable Siegel-theta function.

and $d\mu(\tau) = \frac{du \, dv}{\sqrt{2}}$ if $\tau = u + iv$
The proof uses 2 ingredients:

1. \[ \Delta_z \Theta(z, w) = \Delta_w \Theta(z, w) \]

2. reproducing property of \( G_s^{Y_j}(z, w) \) for \( j = 1, 2 \)