A Classification of Rational Isogeny-Torsion Graphs

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A rational elliptic curve, $E/\mathbb{Q}$, is a smooth projective curve of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

for some $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q}$ with a point at infinity defined over $\mathbb{Q}$, $O = [0 : 1 : 0]$. 

**Theorem (Mordell–Weil, 1922)**

Let $E/\mathbb{Q}$ be an elliptic curve. Then $E(\mathbb{Q})$ is a finitely generated abelian group, i.e., $E(\mathbb{Q})_{\text{tors}}$ is finite abelian and $E(\mathbb{Q}) \cong \mathbb{Z}_{R_E} \times E(\mathbb{Q})_{\text{tors}}$.

**Theorem (Mazur, 1978)**

$E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of the following groups $\mathbb{Z}/M\mathbb{Z}$ with $1 \leq M \leq 10$ or $M = 12$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with $1 \leq N \leq 4$. 
**Elliptic Curves**

**Definition**

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Definition

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Theorem (Mordell–Weil, 1922)

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$$\mathbb{Z}/M\mathbb{Z} \text{ with } 1 \leq M \leq 10 \text{ or } M = 12$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} \text{ with } 1 \leq N \leq 4$$
Isogenies

Definition
Let $E/\mathbb{Q}$ and $E'/\mathbb{Q}$ be elliptic curves. An **isogeny** mapping $E$ to $E'$ is a morphism $\phi: E \to E'$ such that $\phi(O_E) = O_{E'}$. $E$ and $E'$ are said to be **isogenous** if there exists a nonconstant isogeny from $E$ to $E'$. The set of all elliptic curves isogenous to $E$ is called the **isogeny class of $E$**.

Definition
Let $E/\mathbb{Q}$ be a rational elliptic curve. The **isogeny graph** of $E$ is a visualization of the isogeny class of $E$ with edges being rational isogenies generated by the finite cyclic $\mathbb{Q}$-rational subgroups of $E$ and vertices being pairwise non-isomorphic rational elliptic curves isogenous to $E$ that are generated by the finite cyclic $\mathbb{Q}$-rational subgroups of $E$. 
Let \( E/\mathbb{Q} : y^2 + xy + y = x^3 - x^2 - 6x - 4 \) with LMFDB label 17.a2. Then the following is the rational isogeny graph of \( E \):
Mazur’s theorem establishes the possibilities for $E(\mathbb{Q})_{\text{tors}}$.

**Question:** What are the possibilities for torsion at every vertex of isogeny graph?
Mazur’s theorem establishes the possibilities for $E(\mathbb{Q})_{\text{tors}}$.

**Question**: What are the possibilities for torsion at every vertex of isogeny graph?

Let $E/\mathbb{Q} : y^2 + xy + y = x^3 - x^2 - 6x - 4$. Then the following are the rational isogeny graph and the rational isogeny-torsion graph of $E$:
More Examples of Isogeny-Torsion Graphs
Is there an example of the following rational isogeny-torsion graph?

\[
\begin{array}{c}
\mathbb{Z}/4\mathbb{Z} \\
\uparrow \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\downarrow \\
\mathbb{Z}/4\mathbb{Z} & \quad \quad & \mathbb{Z}/4\mathbb{Z}
\end{array}
\]
Is there an example of the following rational isogeny-torsion graph?

Answer: No!
Can we classify ALL rational isogeny-torsion graphs?

In other words, can we classify the size and shape of a rational isogeny graph and the torsion groups of its vertices?
Can we classify ALL rational isogeny-torsion graphs?

In other words, can we classify the size and shape of a rational isogeny graph \textit{and} the torsion groups of its vertices?

\textbf{Theorem (C., Lozano-Robledo)}

\textit{There are at least 37 and at most 39 possible rational isogeny-torsion graphs.}
Theorem (B. Mazur, 1978)

Let $E/\mathbb{Q}$ be an elliptic curve. A prime degree $\mathbb{Q}$-rational isogeny of $E$ has degree $2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, \text{ or } 163$.

Theorem (M. Kenku, 1982)

Let $E/\mathbb{Q}$ be an elliptic curve. Then there are at most $8$ pairwise non-isomorphic rational elliptic curves that are isogenous to $E$.

Note: There is no analogy to Mazur’s or Kenku’s theorems for higher degree number fields. $\mathbb{Q}$ is the only number field over which we can classify isogeny-torsion graphs. Mazur’s and Kenku’s theorems give us a classification of the sizes and shapes of all rational isogeny graphs. They are one of the following:
**$L_k$ Graphs**

**Linear** graphs with $k = 1, 2, 3, \text{ or } 4$ vertices.

\{O\}

Isogeny Class 37.a

Isogeny Class 121.a

Isogeny Class 11.a

Isogeny Class 432.e

(Images courtesy of the LMFDB.)
$R_k$ Graphs

$R_k$: **Rectangular** graphs with $k = 4$ or $6$ vertices.

(Isogeny Class 66.c)

(Isogeny Class 14.a)

(Images courtesy of the LMFDB.)
$T_4$ graphs

$T_4$: Graphs with a single elliptic curve with full two-torsion

Isogeny Class 17.a

(Image courtesy of the LMFDB.)
$T_6$ graphs

$T_6$: Graphs with two rational elliptic curves with full two-torsion and no 3-isogenies

Isogeny Class 21.a

(Image courtesy of the LMFDB.)
$T_8$ graphs

$T_8$: Graphs with three rational elliptic curves with full two-torsion

Isogeny Class 210.e

(Image courtesy of the LMFDB.)
S: Graphs with two rational elliptic curves with full two-torsion and a 3-isogeny

Isogeny Class 30.a

(Image courtesy of the LMFDB.)
## Classification of all $L_k$ Graphs

For the following, we abbreviate $\mathbb{Z}/a\mathbb{Z} = [a]$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} = [2, b]$

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>Isomorphism Types</th>
<th>LMFDB Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>$([1])$</td>
<td>37.a</td>
</tr>
<tr>
<td></td>
<td>$([1],[1])$</td>
<td>75.c</td>
</tr>
<tr>
<td></td>
<td>$([2],[2])$</td>
<td>46.a</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$([3],[1])$</td>
<td>44.a</td>
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<td></td>
<td>$([5],[1])$</td>
<td>38.b</td>
</tr>
<tr>
<td></td>
<td>$([7],[1])$</td>
<td>26.b</td>
</tr>
<tr>
<td></td>
<td>$([1],[1],[1])$</td>
<td>99.d</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$([3],[3],[1])$</td>
<td>19.a</td>
</tr>
<tr>
<td></td>
<td>$([5],[5],[1])$</td>
<td>11.a</td>
</tr>
<tr>
<td></td>
<td>$([9],[3],[1])$</td>
<td>54.b</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$([1],[1],[1],[1])$</td>
<td>432.e</td>
</tr>
<tr>
<td></td>
<td>$([3],[3],[3],[1])$</td>
<td>27.a</td>
</tr>
</tbody>
</table>

Table 2. The list of all $L_k$ rational-isogeny graphs
## Classification of all $R_k$ Graphs

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>Isomorphism Types</th>
<th>LMFDB Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_4$</td>
<td>$([1],[1],[1],[1])$</td>
<td>400.f</td>
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<tr>
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<td>$([2],[2],[2],[2])$</td>
<td>49.a</td>
</tr>
<tr>
<td></td>
<td>$([3],[3],[1],[1])$</td>
<td>50.a</td>
</tr>
<tr>
<td></td>
<td>$([5],[5],[1],[1])$</td>
<td>50.b</td>
</tr>
<tr>
<td></td>
<td>$([6],[6],[2],[2])$</td>
<td>20.a</td>
</tr>
<tr>
<td></td>
<td>$([10],[10],[2],[2])$</td>
<td>66.c</td>
</tr>
<tr>
<td>$R_6$</td>
<td>$([2],[2],[2],[2],[2],[2])$</td>
<td>98.a</td>
</tr>
<tr>
<td></td>
<td>$([6],[6],[6],[6],[2],[2])$</td>
<td>14.a</td>
</tr>
</tbody>
</table>

Table 4. The list of all $R_k$ rational-isogeny graphs
## Classification of all $T_k$ Graphs

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>Isomorphism Types</th>
<th>LMFDB Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_4$</td>
<td>$([2,2], [2], [2], [2])$</td>
<td>120.a</td>
</tr>
<tr>
<td></td>
<td>$([2,2], [2], [4], [2])$</td>
<td>33.a</td>
</tr>
<tr>
<td></td>
<td>$([2,2], [2], [4], [4])$</td>
<td>17.a</td>
</tr>
<tr>
<td>$T_6$</td>
<td>$([2,4],[4],[4],[2,2],[2],[2])$</td>
<td>24.a</td>
</tr>
<tr>
<td></td>
<td>$([2,4],[8],[4],[2,2],[2],[2])$</td>
<td>21.a</td>
</tr>
<tr>
<td></td>
<td>$([2,2],[2],[2],[2,2],[2],[2])$</td>
<td>126.a</td>
</tr>
<tr>
<td></td>
<td>$([2,2],[4],[2],[2,2],[2],[2])$</td>
<td>63.a</td>
</tr>
<tr>
<td>$T_8$</td>
<td>$([2,8],[8],[8],[2,4],[4],[2,2],[2],[2])$</td>
<td>210.c</td>
</tr>
<tr>
<td></td>
<td>$([2,4],[4],[4],[2,4],[4],[2,2],[2],[2])$</td>
<td>195.a</td>
</tr>
<tr>
<td></td>
<td>$([2,4],[4],[4],[2,4],[8],[2,2],[2],[2])$</td>
<td>15.a</td>
</tr>
<tr>
<td></td>
<td>$([2,4],[8],[4],[2,4],[4],[2,2],[2],[2])$</td>
<td>1230.f</td>
</tr>
<tr>
<td></td>
<td>$([2,2],[2],[2],[2,2],[2],[2,2],[2],[2])$</td>
<td>45.a</td>
</tr>
<tr>
<td></td>
<td>$([2,2],[4],[2],[2,2],[2],[2,2],[2],[2])$</td>
<td>75.b</td>
</tr>
</tbody>
</table>

**Table 3.** The list of all $T_k$ rational-isogeny graphs
## Classification of all $S$ Graphs

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>Isomorphism Types</th>
<th>LMFDB Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>([2,2],[2],[2],[2],[2],[2],[2],[2])</td>
<td>240.b</td>
</tr>
<tr>
<td></td>
<td>([2,2],[2],[2],[4],[2,2],[2],[2],[4])</td>
<td>150.b</td>
</tr>
<tr>
<td></td>
<td>([2,2],[4],[4],[2,2],[2],[4],[4])</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>([2,6],[6],[6],[6],[2,2],[2],[2],[2])</td>
<td>30.a</td>
</tr>
<tr>
<td></td>
<td>([2,6],[6],[6],[12],[2,2],[2],[2],[4])</td>
<td>90.c</td>
</tr>
<tr>
<td></td>
<td>([2,2],[6],[12],[12],[2,2],[2],[4],[4])</td>
<td>X</td>
</tr>
</tbody>
</table>

**Table 5.** The list of all (possible) $S$ rational-isogeny graphs
Examples of 21-isogenies

Let $E/\mathbb{Q}$ be an elliptic curve with a finite cyclic $\mathbb{Q}$-rational group of order 21. Then there exist examples of the following rational isogeny-torsion graphs:

Isogeny Class 162.b

\[
\begin{array}{c}
\mathbb{Z}/3\mathbb{Z} \rightarrow \mathcal{O} \\
\downarrow \quad \downarrow \\
\mathbb{Z}/3\mathbb{Z} \rightarrow \mathcal{O}
\end{array}
\]

Isogeny Class 1296.f

\[
\begin{array}{c}
\mathcal{O} \rightarrow \mathcal{O} \\
\downarrow \quad \downarrow \\
\mathcal{O} \rightarrow \mathcal{O} \\
\downarrow \quad \downarrow \\
\mathcal{O} \rightarrow \mathcal{O}
\end{array}
\]
Non-examples of 21-isogenies

The following rational isogeny-torsion graphs do not occur.

\[
\begin{array}{ccc}
\mathbb{Z}/3\mathbb{Z} & \rightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O} & \rightarrow & \mathbb{Z}/3\mathbb{Z}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{Z}/7\mathbb{Z} & \rightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathbb{Z}/7\mathbb{Z} & \rightarrow & \mathcal{O}
\end{array}
\]
The following two examples of rational isogeny-torsion graphs with 27-isogenies exist.

\[
\mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow O
\]

LMFDB Label 27.a

\[
O \rightarrow O \rightarrow O \rightarrow O \rightarrow O
\]

LMFDB Label 432.e

Reasoning: All rational 27-isogenies are CM corresponding to one \( j \)-invariant and no twists of this curve produce this graph.
The following two examples of rational isogeny-torsion graphs with 27-isogenies exist.

\[
\mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow O
\]
LMFDB Label 27.a

\[
O \rightarrow O \rightarrow O \rightarrow O \rightarrow O
\]
LMFDB Label 432.e

The following rational isogeny-torsion graph does not occur.

\[
\mathbb{Z}/9\mathbb{Z} \rightarrow \mathbb{Z}/9\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow O
\]

Reasoning: All rational 27-isogenies are CM corresponding to one \(j\)-invariant and no twists of this curve produce this graph.
Let $E/\mathbb{Q}$ be an elliptic curve. Suppose $E$ has 4 curves in its isogeny class and

$$E(\mathbb{Q})_{\text{tors}} = E[2] = \langle P, Q \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ 

What are the possible isogeny-torsion graphs of $E$?

- Finite cyclic $\mathbb{Q}$-rational subgroups of $E$ are $\{O\}, \langle P \rangle, \langle Q \rangle$ and $\langle P + Q \rangle$.
- $(E/\langle P \rangle)(\mathbb{Q})_{\text{tors}}, (E/\langle Q \rangle)(\mathbb{Q})_{\text{tors}}$, and $(E/\langle P + Q \rangle)(\mathbb{Q})_{\text{tors}}$ are cyclic.
- $E$ has a point of order 2 defined over $\mathbb{Q}$, thus all isogenous curves do too, but because $C(E) = 4$, no curve can have a point of order 8 defined over $\mathbb{Q}$. No points of odd order defined over $\mathbb{Q}$. 

![Diagram of isogeny-torsion graphs]
Let’s assume the following isogeny-torsion graph exists.

\[
\begin{array}{c}
\mathbb{Z}/4\mathbb{Z} \\
\uparrow \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\downarrow & \downarrow \\
\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z}
\end{array}
\]
Classification of $T_4$ Graphs (3)

Assume $E$ is non-CM and $(E/\langle P \rangle)(\mathbb{Q})_{\text{tors}}, (E/\langle Q \rangle)(\mathbb{Q})_{\text{tors}},$ and $(E/\langle P + Q \rangle)(\mathbb{Q})_{\text{tors}},$ are cyclic of order 4. Then the image of the mod 4 Galois representation of $E$ is conjugate to

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \right\} \in GL_2(\mathbb{Z}/4\mathbb{Z})$$

but no group in the RZB database of images of 2-adic Galois representations of rational non-CM elliptic curves reduces mod 4 to this group.
Assume $E$ is non-CM and $(E/⟨P⟩)(\mathbb{Q})_{\text{tors}}, (E/⟨Q⟩)(\mathbb{Q})_{\text{tors}},$ and $(E/⟨P + Q⟩)(\mathbb{Q})_{\text{tors}},$ are cyclic of order 4. Then the image of the mod 4 Galois representation of $E$ is conjugate to

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \right\} \in GL_2(\mathbb{Z}/4\mathbb{Z})$$

but no group in the RZB database of images of 2-adic Galois representations of rational non-CM elliptic curves reduces mod 4 to this group.

Suppose $E$ is CM. Then there are only finitely many $j$-invariants that correspond to a torsion subgroup with full two-torsion. No quadratic twist will give you an isogeny-torsion graph with all three $(E/⟨P⟩)(\mathbb{Q})_{\text{tors}}, (E/⟨Q⟩)(\mathbb{Q})_{\text{tors}},$ and $(E/⟨P + Q⟩)(\mathbb{Q})_{\text{tors}},$ cyclic of order 4.
Classification of $T_4$ Graphs (3)

- Assume $E$ is non-CM and $(E/\langle P \rangle)(\mathbb{Q})_{\text{tors}}, (E/\langle Q \rangle)(\mathbb{Q})_{\text{tors}}$, and $(E/\langle P + Q \rangle)(\mathbb{Q})_{\text{tors}}$, are cyclic of order 4. Then the image of the mod 4 Galois representation of $E$ is conjugate to

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \right\} \in GL_2(\mathbb{Z}/4\mathbb{Z})$$

but no group in the RZB database of images of 2-adic Galois representations of rational non-CM elliptic curves reduces mod 4 to this group.

- Suppose $E$ is CM. Then there are only finitely many $j$-invariants that correspond to a torsion subgroup with full two-torsion. No quadratic twist will give you an isogeny-torsion graph with all three $(E/\langle P \rangle)(\mathbb{Q})_{\text{tors}}, (E/\langle Q \rangle)(\mathbb{Q})_{\text{tors}}$, and $(E/\langle P + Q \rangle)(\mathbb{Q})_{\text{tors}}$, cyclic of order 4.

- Isogeny classes with LMFDB labels 120.a, 33.a, and 17.a correspond to $T_4$ isogeny graphs with zero, one, and two point-wise rational groups of order 4 respectively.
All $T_4$ Graphs

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/4\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/4\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/4\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z}$
Graphs Not Yet Ruled Out

\[
\begin{align*}
\mathbb{Z}/6\mathbb{Z} & \rightarrow \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} & \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/12\mathbb{Z} & \rightarrow \mathbb{Z}/4\mathbb{Z} \\
\mathbb{Z}/2\mathbb{Z} & \rightarrow \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/4\mathbb{Z} & \rightarrow \mathbb{Z}/4\mathbb{Z} \\
\mathbb{Z}/4\mathbb{Z} & \rightarrow \mathbb{Z}/4\mathbb{Z} \\
\mathbb{Z}/12\mathbb{Z} & \rightarrow \mathbb{Z}/4\mathbb{Z}
\end{align*}
\]
The image of the mod 4 Galois representations of the two unconfirmed graphs are conjugate to

\[ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \right\} \in GL_2(\mathbb{Z}/4\mathbb{Z}) \]

Find the image in RZB database and get its $j$-invariant.

Add a 3-isogeny to these images by comparing it to $j$-invariant of a curve with a 3-isogeny

This defines a curve of genus 1, 3, or 7. And we have not been able to find all rational points of those curves as of yet.
Questions?