Generalised Heegner cycles and Griffiths groups of infinite rank

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Maine-Québec Number Theory Conference

October 5, 2019

Let X be a smooth projective variety over a field k.

Definition

An algebraic cycle of codimension r is a formal finite sum

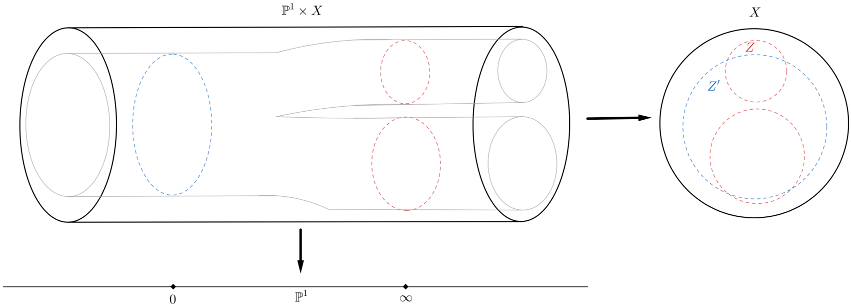
$$\sum_{Z \subset X} n_Z \cdot Z$$

where Z is a subvariety of X of codimension r and $n_Z \in \mathbf{Z}$. The set of such objects forms a group denoted $\mathcal{Z}^r(X)(k)$.

Example (E is an elliptic curve over $\mathbf{Q}$)

$$\mathcal{Z}^1(E)(\bar{\mathbf{Q}}) = \mathrm{Div}(E) = \left\{ \sum_{P \in E(\bar{\mathbf{Q}})} n_P \cdot P \ : \ n_P \in \mathbf{Z}, \ \text{finite sum} \right\}$$

Rational and Algebraic Equivalence



Homological Equivalence

$$\mathcal{Z}^r(X)(k)_0 := \ker(\mathcal{Z}^r(X)(k) \xrightarrow{\mathrm{cl}_p} H^{2r}_{\mathrm{et}}(X_{\bar{k}}, \mathbf{Q}_p)(r)^{G_k}).$$

(independent of p if $\mathrm{char}(k) = 0$)

$$\mathcal{Z}^r(X)(k)_{\mathrm{rat}} \subset \mathcal{Z}^r(X)(k)_{\mathrm{alg}} \subset \mathcal{Z}^r(X)(k)_0 \subset \mathcal{Z}^r(X)(k)$$

By taking the quotient by $\mathcal{Z}^r(X)(k)_{\mathrm{rat}}$ one gets the associated filtration of the Chow group:

$$0 \subset \mathrm{CH}^r(X)(k)_{\mathrm{alg}} \subset \mathrm{CH}^r(X)(k)_0 \subset \mathrm{CH}^r(X)(k).$$

The graded piece $\mathrm{Gr}^r(X)(k) := \mathrm{CH}^r(X)(k)_0 / \mathrm{CH}^r(X)(k)_{\mathrm{alg}}$ is called the Griffiths group.

Example (E is an elliptic curve)

- $\mathrm{CH}^1(E)(\bar{\mathbf{Q}}) = \mathrm{Div}(E)/P(E) = \mathrm{Pic}(E)$
- $\mathrm{CH}^1(E)(\bar{\mathbf{Q}})_0 = \mathrm{Pic}^0(E) = E(\bar{\mathbf{Q}})$
- $\mathrm{CH}^1(E)(\bar{\mathbf{Q}})_{\mathrm{alg}} = \mathrm{Pic}^0(E) = E(\bar{\mathbf{Q}})$
- $\mathrm{Gr}^1(E)(\bar{\mathbf{Q}}) = 0.$

Let X be a smooth projective variety over a number field K. For each $n \geq 0$, one can associate to X a complex L-function

$$L(H^n_{\mathrm{et}}(X_{\bar{K}}), s).$$

Beilinson-Bloch Conjecture

For each $0 \leq j \leq \dim(X)$, $\mathrm{CH}^j(X)(K)_0$ is a finitely generated abelian group and

$$\dim_{\mathbf{Q}} \mathrm{CH}^j(X)(K)_0 \otimes \mathbf{Q} = \mathrm{ord}_{s=j} L(H^{2j-1}_{\mathrm{et}}(X_{\bar{K}}), s).$$

In the case of an elliptic curve E over $\mathbf{Q}$, this is the Birch and Swinnerton-Dyer conjecture

$$\mathrm{rank}(E(\mathbf{Q})) = \mathrm{ord}_{s=1} L(E/\mathbf{Q}, s).$$

Let X be a smooth projective variety over $\mathbf{C}$. One can define the complex cycle class map

$$\begin{array}{rccc} \mathrm{cl}_{\mathbf{C}}: & \mathrm{CH}^r(X)(\mathbf{C}) & \longrightarrow & H^{2r}(X(\mathbf{C}),\mathbf{Z}). \\ & [Z] & \mapsto & \left(\omega \mapsto \int_Z \omega\right). \end{array}$$

The image of this map lies in the subgroup of Hodge classes

$$\mathrm{Hdg}^{2r}(X(\mathbf{C})) := H^{2r}(X(\mathbf{C}),\mathbf{Z}) \cap H^{r,r}(X(\mathbf{C})).$$

Hodge Conjecture

The image of $\mathrm{cl}_{\mathbf{C}} \otimes \mathbf{Q}$ is equal to $\mathrm{Hdg}^{2r}(X(\mathbf{C})) \otimes \mathbf{Q}$.

The Tate conjecture is the arithmetic analog of the Hodge conjecture and is concerned with the p-adic étale cycle class map.

Recall the Griffiths group

$$\mathrm{Gr}^j(X)(k) = \mathrm{CH}^j(X)(k)_0 / \mathrm{CH}^j(X)(k)_{\mathrm{alg}}.$$

- Griffiths ('69), Clemens ('83) and Ceresa ('83): first results - transcendental methods over **C**.
- Harris ('83) and Bloch ('84): first example of non-triviality for varieties over number fields - the Ceresa cycle on the Fermat quartic $T_0^4 + T_1^4 = T_2^4$.
- Schoen ('86): infinite rank over $\bar{\mathbf{Q}}$ for Kuga-Sato threefold using Heegner cycles.
- BDP ('17): non-torsion elements using generalised Heegner cycles.

- K = imaginary quadratic field with ring of integers $\mathcal{O}_K$, satisfying Heegner hypothesis
- H = Hilbert class field of K
- A = elliptic curve over H with $\mathrm{End}_H(A) \cong \mathcal{O}_K$, $A(\mathbf{C}) = \mathbf{C}/\mathcal{O}_K$
- $W_r = r^{\text{th}}$ Kuga-Sato variety over $X_1(N)$.
- $W_r \times A^r$ smooth proper variety over H of dimension $2r+1$, naturally fibered over $X_1(N)$, with fibre over an elliptic curve E equal to $E^r \times A^r$.

Definition

Generalised Heegner cycles are a distinguished collection of cycles

$$\Delta_\varphi \in \mathrm{CH}^{r+1}(W_r \times A^r)(F_\varphi)_0$$

indexed by $\varphi \in \mathrm{Isog}^{\mathfrak{N}}(A)$ with F_φ a finite extension of H.

Theorem (Bertolini-Darmon-L.-Prasanna '19)

For all $r \geq 0$,

$$\dim_{\mathbf{Q}} \mathrm{CH}^{r+1}(W_r \times A^r)(\bar{\mathbf{Q}})_0 \otimes \mathbf{Q} = \infty.$$

Furthermore, for all $r \geq 2$,

$$\dim_{\mathbf{Q}} \mathrm{Gr}^{r+1}(W_r \times A^r)(\bar{\mathbf{Q}}) \otimes \mathbf{Q} = \infty.$$

Theorem (Schoen '86)

$$\dim_{\mathbf{Q}} \mathrm{Gr}^2(W_2)(\bar{\mathbf{Q}}) \otimes \mathbf{Q} = \infty.$$

The proof is an adaptation of the arguments of Schoen to the setting of generalised Heegner cycles.

1. Massage the variety and the cycles using algebraic idempotent correspondences.
2. Complex analytic and Hodge theoretic arguments involving the complex Abel-Jacobi map.
3. Arithmetic input using étale cohomology.
4. Linear independence using Galois theory.

Thank you for your attention !