On the classification of rigid meromorphic cocycles

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The setup

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- Let $\Gamma := \text{SL}_2(\mathbb{Z}[1/p])$ and let $\mathcal{M}^\times$ be the multiplicative group of rigid meromorphic functions on $\mathcal{H}_p$.  

A rigid meromorphic cocycle is a class in $H^1_f(\Gamma, \mathcal{M}^\times)$, i.e. a class assuming constant values on $\text{Stab}(\infty)$.

The values of these cocycles at $\mathcal{M}$ points were studied by Darmon and Vonk in the paper *Singular moduli for real quadratic fields: a rigid analytic approach*.
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- A \textit{rigid meromorphic cocycle} is a class in $H^1_f(\Gamma, \mathcal{M}^\times)$, i.e. a class assuming constant values on $\text{Stab}(\infty)$.
- The values of these cocycles at RM points were studied by Darmon and Vonk in the paper \textit{Singular moduli for real quadratic fields: a rigid analytic approach}.
The values at RM points

- Let $\tau$ be an RM point on $\mathcal{H}_p$ and $\mathcal{O}_\tau := \mathbb{Z}\tau + \mathbb{Z}$.
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- The map $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto c\tau + d$ gives an isomorphism between $\text{Stab}(\tau)$ and $(\mathcal{O}_\tau[1/p])^1$, the group of norm one units in $\mathcal{O}_\tau[1/p]^{\times}$. 

- Conjecture (Darmon, Vonk) $J[\tau]$ is an algebraic number in $H_\tau \cdot H_J$, where $H_\tau$ is the narrow ring class field associated to $\mathcal{O}_\tau$ and $H_J$ is the compositum of the fields $H_\tau$ for $j(\tau) = \infty$. 

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- Hence $\text{Stab}(\tau)$ is generated by a fundamental unit $\gamma_\tau$.
- Given $J \in H^1_f(\Gamma, M^\times)$ we define $J[\tau] := J(\gamma_\tau)(\tau)$.
- Let $j$ be a rigid meromorphic period function associated to $J$. 

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The values at RM points

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- The map \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto c\tau + d \) gives an isomorphism between \( \text{Stab}(\tau) \) and \( (\mathcal{O}_\tau[1/p])^1 \), the group of norm one units in \( \mathcal{O}_\tau[1/p]^\times \).
- Hence \( \text{Stab}(\tau) \) is generated by a fundamental unit \( \gamma_\tau \).
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Classification of the cocycles

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- Let $\mathcal{M}_2$ be the additive group $\mathcal{M}$ with the weight two action of $\Gamma$:
  $$f|_2 \gamma = (c\tau + d)^{-2} f(\gamma \tau) \text{ where } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
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A *rigid meromorphic cocycle of weight two* is a class in $H^1_{\text{par}}(\Gamma, \mathcal{M}_2)$, i.e. vanishing on $\text{Stab}(\infty)$. 
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Hence we first compute $\mathbb{H}^1_{\text{par}}(\Gamma, \mathcal{M}_2)$.

Inspiration comes from the classification of $\mathbb{H}^1_{\text{par}}(\text{PSL}_2(\mathbb{Z}), M)$, where $M$ are rational functions (Choie, Zagier).
A rational period function (RPF) for $\text{PSL}_2(\mathbb{Z})$ is a rational function $q$ such that $q|_2(1 + T) = 0 = q|_2(1 + U + U^2)$, where $T$ and $U$ are the order 2 and 3 generators of $\text{PSL}_2(\mathbb{Z})$. 
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All poles of RPFs have order 1 and are simple real quadratic irrationalities, so one concludes:
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\textbf{Theorem (Choie, Zagier)}

Any RPF is a linear combination of the functions

$$\frac{1}{z} \quad \text{and} \quad \phi_\tau(z) = \sum \text{sgn}(\omega) \frac{1}{z - \omega},$$

where $\omega \in \text{PSL}_2(\mathbb{Z})_\tau$ for $\tau$ ranging through $\text{PSL}_2(\mathbb{Z})$-representatives of simple real quadratic irrationalities.
Similarly one can define *rigid meromorphic period functions* (RMPF) and use them to classify $H^1_{par}(\Gamma, \mathcal{M}_2)$, getting:

**Theorem (Darmon, Vonk)**

Any RMPF is a linear combination of a rigid analytic period function and of the functions

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where $\tau \in \Gamma \setminus \text{RM}_p$, $\omega \in \Gamma \tau$ and $\omega$ is simple.

Using the logarithmic derivative one gets:

**Theorem (Darmon, Vonk)**

For all primes $p$, the group $H^1_{f}(\Gamma, \mathcal{M}_2)$ is of infinite rank over $\mathbb{Z}$.
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Back to the classification of the cocycles

- Similarly one can define *rigid meromorphic period functions* (RMPF) and use them to classify $H_{par}^1(\Gamma, \mathcal{M}_2)$, getting:

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where $\tau \in \Gamma \setminus \mathcal{H}_p^{RM}$, $\omega \in \Gamma \tau$ and $\omega$ is simple.

- Using the logarithmic derivative one gets:

**Theorem (Darmon, Vonk)**

For all primes $p$, the group $H_f^1(\Gamma, \mathcal{M}^\times)$ is of infinite rank over $\mathbb{Z}$. 
We can ask what happens if $\Delta$ is a congruence subgroup of $\Gamma$, for example $\Delta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \ c = 0 \ (\text{mod} \ q), \ q \neq p \ \text{prime} \right\}$. 

Using RPFs is probably not the right approach anymore. Moreover, the sum used to define $\psi_\tau(z)$ does not converge anymore (the intersection of $\Delta_\tau$ and any affinoid does not have the same number of positive and negative elements). A possible source of inspiration might be the work of Ash, who classified $H_1^{\text{par}}(G, M)$, where $G$ is any congruence subgroup of $\text{SL}_2(\mathbb{Z})$. 

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We still have $d\log: H^1_f(\Delta, \mathcal{M}^\times) \to H^1_{\text{par}}(\Delta, \mathcal{M}_2)$. Using RPFs is probably not the right approach anymore. Moreover, the sum used to define $\psi_\tau(z)$ does not converge anymore (the intersection of $\Delta_\tau$ and any affinoid does not have the same number of positive and negative elements).

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Generalization

- We can ask what happens if $\Delta$ is a congruence subgroup of $\Gamma$, for example $\Delta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \; c \equiv 0 \pmod{q}, \; q \neq p \text{ prime} \right\}$.
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Thank you!