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Primitive Sets

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A subset $\mathcal{A}$ of the positive integers is said to be *primitive* if no member of $\mathcal{A}$ divides another.

Some examples:

1. The set $\mathcal{P}$ of prime numbers.

2. More generally, the set $\mathbb{N}_k = \{n : \Omega(n) = k\}$, where $\Omega(n)$ is the number of prime factors of n counted with repetition.

3. The set $(x, 2x] \cap \mathbb{N}$.

4. With σ the sum-of-divisors function, the set

$$\mathcal{A} = \{n \in \mathbb{N} : \sigma(n)/n \geq 2,\ \sigma(d)/d < 2 \text{ for all } d \mid n,\, d < n\}.$$

This last example goes back to **Pythagoras**.

He was quite interested in the sum-of-divisors function σ, and he and his followers classified the natural numbers into 3 categories:

- *deficient*: $\sigma(n)/n < 2$, like $n = 1, 2, 3, 4, 5, 7, 8, 9, 10, \ldots$,

- *perfect*: $\sigma(n)/n = 2$, like $n = 6, 28, 496, 8128, \ldots$,

- *abundant*: $\sigma(n)/n > 2$, like $n = 12, 18, 20, 24, \ldots$.

The primitive set

$$\mathcal{A} = \{n \in \mathbb{N} : \sigma(n)/n \geq 2,\ \sigma(d)/d < 2 \text{ for all } d \mid n, d < n\}.$$

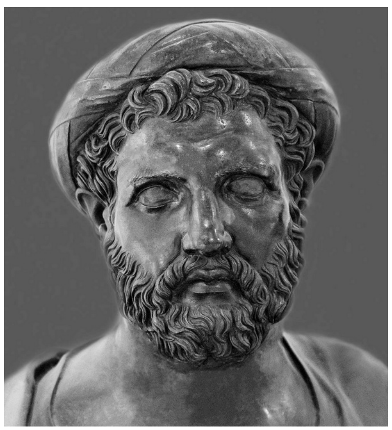
consists of the nondeficient numbers n with every proper divisor deficient. The set of multiples of A consists of all of the nondeficient numbers.

So, if the sum of reciprocals of the members of $\mathcal{A}$ is convergent, then the nondeficient numbers would have an asymptotic density.

Since it's easy to see the perfect numbers have density 0 (Euler showed that any perfect number is of the form pm^2 where p is a prime factor of $\sigma(m^2)$), it would follow that the abundant numbers have a density, as do the deficient numbers.

Erdős (1934). *The reciprocal sum of the primitive nondeficient numbers is finite.*

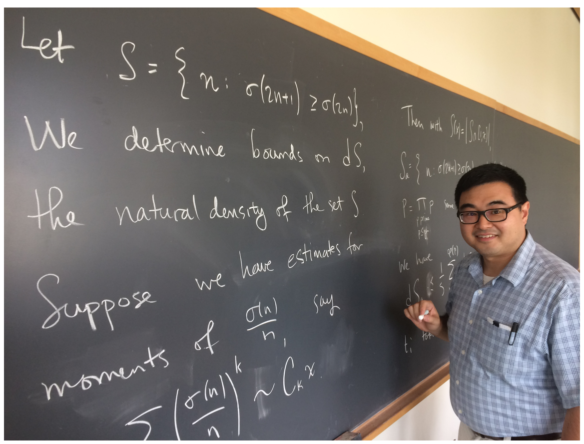
Corollary. *The set of nondeficient numbers has a positive density.*



Pythagoras of Samos   Leonhard Euler   Paul Erdős

We now know (**Kobayashi**, 2010) that the density of the abundant numbers (= the density of the nondeficient numbers) lies between 0.2476171 and 0.2476475.

And we know (**Lichtman**, 2018) that the sum of reciprocals of the primitive nondeficient numbers is between 0.34842 and 0.37937.



Mitsuo Kobayashi

Jared Lichtman

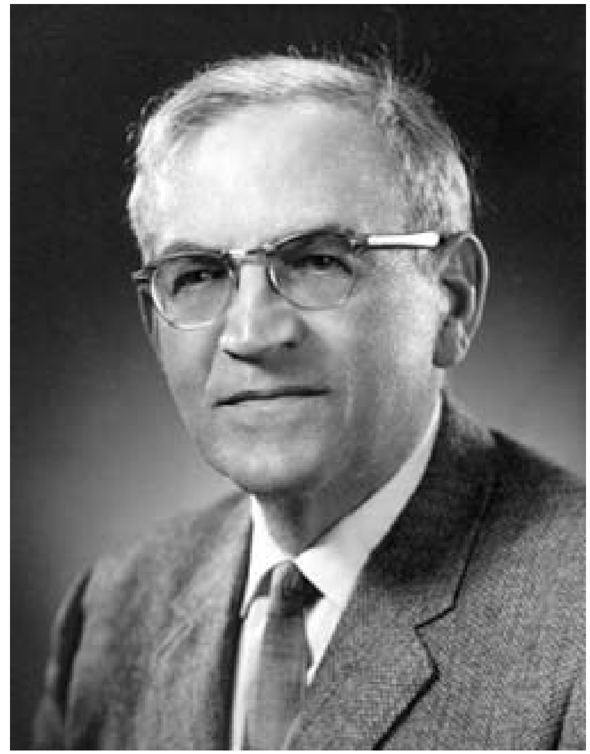
Actually, it was known before **Erdős** that the density of the nondeficient numbers exists:

Davenport (1933) showed the density $D(u)$ of $\{n : \sigma(n)/n \le u\}$ exists, and that $D(u)$ is continuous.

Davenport strongly used a technical criterion of **Schoenberg**, who in 1928 proved an analogous result for the density of numbers n with $n/\varphi(n) \le u$. Here φ is Euler's function.

With his paper on primitive nondeficient numbers in 1934, **Erdős** began studying this subject, looking for the great theorem that would unite these threads. His elementary approach through primitive sets led him to believe that non-technical methods could be used.

Harold Davenport



Isaac J. Schoenberg

This culminated in the **Erdős–Wintner** theorem in 1939 and the **Erdős–Kac** theorem the same year. And so was born the subject of probabilistic number theory.

Erdős became interested in primitive sets for their own sake, and this led in interesting directions as well.

Aurel Wintner

Mark Kac

One might guess that a primitive set always has asymptotic density 0. It's true for our 4 examples, and more generally, it's true that the lower asymptotic density of a primitive set must be 0. Somewhat counter-intuitively, the upper asymptotic density need not be zero!

In 1934, **Besicovitch** showed that there are primitive sets with upper density arbitrarily close to 1/2. The idea: use the sets $(x, 2x]$ for a rapidly increasing sequence of numbers x, discarding the few numbers from an interval that are multiples of numbers from a prior interval. The fact that there are only a few numbers to discard is the key here.

This lemma, that when x is large, few numbers have a divisor in $(x, 2x]$, led to the "multiplication table theorem" of **Erdős**. But that's a story for a different lecture ... .

The famous **Erdős Conjecture** on primitive sets is based on the following old theorem.

Theorem (Erdős, 1935). *If $\mathcal{A}$ is a primitive set, then*

$$f(\mathcal{A}) := \sum_{\substack{a \in \mathcal{A} \\ a > 1}} \frac{1}{a \log a} < \infty.$$

In fact, $f(\mathcal{A})$ is uniformly bounded as $\mathcal{A}$ varies over primitive sets.

Assume $1 \notin \mathcal{A}$. Let $P(\mathcal{A})$ denote the set of primes dividing some member of $\mathcal{A}$.

Conjecture (Erdős, 1988). *For $\mathcal{A}$ primitive, $f(A) \le f(P(\mathcal{A}))$.*

With $\mathcal{P}$ the set of primes, let $C = f(\mathcal{P}) = 1.63661632336\ldots$, the calculation done by **Cohen**.

The Erdős conjecture is equivalent to:

Conjecture (Erdős, 1988). *For $\mathcal{A}$ primitive,* $f(\mathcal{A}) \le C = 1.63661632336\ldots$, *where* $f(\mathcal{A}) = \sum_{a \in \mathcal{A}} 1/(a \log a)$.

What do we know about $f(\mathcal{A})$?

Erdős, Zhang (unpublished): $f(\mathcal{A}) < 2.886$.

Robin (unpublished): $f(\mathcal{A}) < 2.77$.

Erdős, Zhang (1993): $f(\mathcal{A}) < 1.84$.

Guy Robin

Zhenxiang Zhang

Recall that $\mathbb{N}_k = \{n : \Omega(n) = k\}$,
$\mathcal{P}(\mathcal{A}) = \{p \text{ prime} : p \text{ divides some member of } \mathcal{A}\}$.

Zhang (1991): *If* $f(\mathcal{A}) \le C$ *if each* $a \in \mathcal{A}$ *has* $\Omega(a) \le 4$.

Zhang (1993): *For each* $k \ge 2$, $f(\mathbb{N}_k) < f(\mathbb{N}_1) = C$.

Banks, Martin (2013): *If* $\sum_{p \in \mathcal{P}(\mathcal{A})} 1/p < 1.7401\ldots$, *then* $f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$.

Banks, Martin (2013): **Conjecture**: $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \ldots$.

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Banks, Martin (2013): **Conjecture**: $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \ldots$.

Lichtman (2019): *The Banks–Martin Conjecture is false.*

Bill Banks Greg Martin

Lichtman, P (2019). *For $\mathcal{A}$ primitive,*

- $f(\mathcal{A}) < e^{\gamma} = 1.78109\ldots$.

- *If no member of $\mathcal{A}$ is divisible by 8, then* $f(\mathcal{A}) < f(\mathcal{P}(\mathcal{A})) + 2.37 \times 10^{-7}$.

- *Assuming RH and LI, there is a set of primes $\mathcal{P}_0$ of relative lower logarithmic density ≥ 0.995 such that $f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$ when $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}_0$. Unconditionally, $\mathcal{P}_0$ contains all of the odd primes up to* $\exp(10^6)$.

Note: The relative lower logarithmic density of a set of primes $\mathcal{P}_0$ is

$$\liminf_{x\to\infty} \frac{1}{\log\log x} \sum_{\substack{p\in\mathcal{P}_0\\ p\le x}} \frac{1}{p}.$$

In a new paper **Lichtman, Martin, & P** show that $\mathcal{P}_0$ has relative lower asymptotic density at least $0.99999973\ldots$.

Notation: For an integer $a \ge 2$, let

$$p(a) := \min\{p \text{ prime} : p \mid a\},$$
$$P(a) := \max\{p \text{ prime} : p \mid a\}.$$

A version of the 1935 Erdős argument:

Let $S_a = \{ba : p(b) \ge P(a)\}$. The asymptotic density of S_a is

$$\delta(S_a) = \frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right).$$

Moreover the sets S_a, as a varies over a primitive set $\mathcal{A}$, are pairwise disjoint. So

$$\sum_{a\in\mathcal{A}} \frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right) = \sum_{a\in\mathcal{A}} \delta(S_a) \le \overline{\delta}\left(\bigcup_{a\in\mathcal{A}} S_a\right) \le 1.$$

But

$$\frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right) \gg \frac{1}{a \log P(a)} \ge \frac{1}{a \log a},$$

so that $f(\mathcal{A}) \ll 1$.

We should be more careful with the step where we say $\prod_{p<P(a)}(1-1/p) \gg 1/\log P(a)$.

Using some modern computations involving the theorem of **Mertens**, we were able to show that $f(\mathcal{A}) < e^\gamma$.

To do better, we followed an idea of **Erdős** and **Zhang**: partition $\mathcal{A}$ by the least prime factor of the elements. Let $\mathcal{A}(q) = \{a \in \mathcal{A} : p(a) = q\}$. We'd love to show that $f(\mathcal{A}(q)) \le 1/(q \log q)$, and this is clear if $q \in \mathcal{A}(q)$.

Say $q \notin \mathcal{A}(q)$. Then

$$f(\mathcal{A}(q)) = \sum_{a\in\mathcal{A}(q)} \frac{1}{a\log a} < \sum_{a\in\mathcal{A}(q)} \frac{e^\gamma}{a} \prod_{p<P(a)} (1-1/p)$$

$$= e^\gamma \sum_{a\in\mathcal{A}(q)} \delta(S_a) \le e^\gamma \overline{\delta}(\cup_{a\in\mathcal{A}(q)} S_a) \le \frac{e^\gamma}{q} \prod_{p<q} (1-1/p).$$

Again: $\mathcal{A}(q) = \{a \in \mathcal{A} : p(a) = q\}$. We'd like to show $f(\mathcal{A}(q)) \le 1/(q \log q)$.
This holds if $\mathcal{A}(q) = \{q\}$. If not, we have

$$f(\mathcal{A}(q)) \le \frac{e^\gamma}{q} \prod_{p<q} (1 - 1/p).$$

We'd be laughing if

$$e^\gamma \prod_{p<q} \left(1 - \frac{1}{p}\right) \le \frac{1}{\log q}.$$

In fact, the famous theorem of **Mertens** says that the left side is asymptotically equal to the right side as $q \to \infty$.

Say a prime q is **Mertens** if

$$e^{\gamma} \prod_{p<q} \left(1 - \frac{1}{p}\right) \le \frac{1}{\log q}.$$

So, if q is Mertens, then $f(\mathcal{A}(q)) \le 1/(q \log q)$ regardless if $q \in \mathcal{A}(q)$ or $q \notin \mathcal{A}(q)$. Thus, if every prime in $\mathcal{P}(\mathcal{A})$ is Mertens, then

$$f(\mathcal{A}) = \sum_{q \in \mathcal{P}(\mathcal{A})} f(\mathcal{A}(q)) \le \sum_{q \in \mathcal{P}(\mathcal{A})} \frac{1}{q \log q} = f(\mathcal{P}(\mathcal{A})).$$

Franz Mertens

A prime q is **Mertens** if $e^\gamma \prod_{p<q}(1 - 1/p) \le 1/\log q$. And, if every prime in $\mathcal{P}(\mathcal{A})$ is Mertens, then $f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$. That is, the Erdős conjecture is true for sets supported on the Mertens primes.

Let's try it out. Is 2 Mertens?

$$\prod_{p<2}\left(1 - \frac{1}{p}\right) = 1, \qquad \frac{1}{e^\gamma \log 2} = 0.81001\ldots.$$

So, 2 is not Mertens. ☹

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Theorem (Lamzouri, 2016). *Assuming RH and LI, the set of real numbers* x *with* $e^\gamma \prod_{p\le x}(1-1/p) < 1/\log x$ *has logarithmic density* $0.99999973\ldots$.

Corollary (Lichtman, Martin, P, 2019). *Assuming RH and LI, the set of Mertens primes has relative logarithmic density* 0.99999973... .

Note that 0.99999973... is the exact same log density that **Rubinstein & Sarnak** found in their famous 1994 paper for the set of x where $\mathrm{li}(x) > \pi(x)$, on assumption of RH and LI, though the two sets are not the same. (For technical reasons, the density of the Mertens race and the density of the π/li race turn out to be the same number.)

When we started investigating primitive sets we had no idea that we would find a connection to "Chebyshev's bias" and "prime number races".

For details and our other results, see our papers:

J. D. Lichtman and C. Pomerance, *The Erdős conjecture for primitive sets*, Proc. Amer. Math. Soc. Ser. B **6** (2019), 1–14.

J. D. Lichtman, G. Martin, and C. Pomerance, *Primes in prime number races*, Proc. Amer. Math. Soc. **147** (2019), 3743–3757.

Thank you