On the total number of prime factors of an odd perfect number

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Is there an odd $N$ such that $\sigma(N) = 2N$?
Throughout this talk $N$ will denote an odd perfect number.
What is known?

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- $N < 2^{4\omega}$ and $\omega \geq 10$. (Nielsen)
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**Theorem**

(Z.) If $N$ is an odd perfect number with $3 \nmid N$, then

$$\Omega \geq \frac{302}{113}\omega - \frac{286}{133}.$$  \hfill (1)

If $N$ is an odd perfect number, with $3|N$, then

$$\Omega \geq \frac{66}{25}\omega - 5.$$  \hfill (2)
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\( \sigma(p^2) = p^2 + p + 1 \), and \( \sigma(p^4) = p^4 + p^3 + p^2 + p + 1 \).
• $\sigma(n)$ is multiplicative and $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$.

• $x^k - 1$ factors as the product of cyclotomic polynomials:

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• $\sigma(p^2) = p^2 + p + 1$, and $\sigma(p^4) = p^4 + p^3 + p^2 + p + 1$.

• If $p|n^2 + n + 1$, then $p \equiv 1 \pmod{3}$ or $p = 3$. Similar statement for $p|n^4 + n^3 + n^2 + n + 1$. 
Central ingredients to Ochem and Rao

- Key insight: Either we have many copies of 3 in the factorization, or we have many primes raised to a power greater than 2.
- Use a system of linear inequalities on the number of prime factors.
- If $p \equiv 1 \pmod{3}$, then $3 | \sigma(p^2)$.

Lemma (Ochem and Rao)

Let $p$, $q$, and $r$ be positive integers. If $p^2 + p + 1 = r$ and $q^2 + q + 1 = 3r$, then $p$ is not an odd prime.
Key insight: Either we have many copies of 3 in the factorization, or we have many primes raised to a power greater than 2.

Use a system of linear inequalities on the number of prime factors.

If \( p \equiv 1 \pmod{3} \), then \( 3 \mid \sigma(p^2) \).

If \( p^2 \mid \mid N \), and \( q \mid \sigma(p^2) \), then either \( q^4 \mid N \) or \( q \) contributes a 3.

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Look at number of primes in \( S \) (set of primes of \( N \) which are raised to the second power), and the number of primes in \( T \) (set of primes which are raised to the fourth power), and \( U \) set of primes raised to higher powers. Keep the special prime and the powers of 3 separate.
Lemma

Let $a$ and $b$ be distinct odd primes and $p$ a prime such that $p|(a^2 + a + 1)$ and $p|(b^2 + b + 1)$. If $a \equiv b \equiv 2 \pmod{3}$, then $p \leq \frac{a+b+1}{5}$. If $a \equiv b \equiv 1 \pmod{3}$, then $p \leq \frac{a+b+1}{3}$.

Forces a large set of distinct primes from $S$. 
Define a Triple Threat as a quadruplet of primes \((x, a, b, c)\) with \(\sigma(a^2), \sigma(b^2)\) and \(\sigma(c^2)\) also prime and

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\sigma(x^2) = \sigma(a^2)\sigma(b^2)\sigma(c^2).
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*No triple threat exists with \(x \equiv a \equiv 1 \pmod{5}\).*

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Primes in \(T\) contribute a lot or we have a lot of info about the primes: Set \(g(x) = x^4 + x^3 + x^2 + x + 1,\) \(f(x) = x^2 + x + 1.\) Then
\[f(g(x)) = (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3).\]
Conjecture

Let $p$ and $q$ be distinct odd primes and let $\Phi_p(x)$ and $\Phi_q(x)$ be the $p$th and $q$th cyclotomic polynomials. Then aside from a finite set of exceptions, at least one of $\Phi_p(\Phi_q(x))$ or $\Phi_q(\Phi_p(x))$ is irreducible.
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- Call an ordered pair of positive integers $(m, n)$ a good pair if $\Phi_m(\Phi_n(x))$ factors over the integers where $\Phi_m$ and $\Phi_n$ are the $m$th and $n$th cyclotomic polynomials. Let $D(t)$ count the number of good pairs with both $m \leq t$ and $n \leq t$.

Conjecture

$$\lim_{t \to \infty} \frac{D(t)}{t^2} = 0.$$
Future work

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- Understanding cyclotomic polynomial composition.
- Generalizing these results (Multiply perfect numbers, Ore harmonic numbers).
- Can we get a better than linear inequality?
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