Riemann-Roch and the trace formula

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- Lefschetz number $L(g) = \mathrm{Tr_s}^{H(X,F)}[g]$.

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- Proof based on a suitable deformation (normal cone, embeddings ...)

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- Explicit evaluation of orbital integrals.

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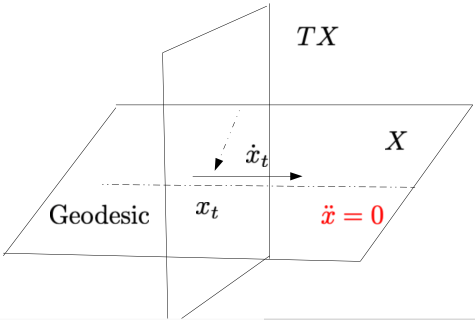
2. Is there a global-local deformation principle?

Geodesics



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Example

$G = \mathrm{SL}_2(\mathbf{R})$, $K = S^1$, X upper half-plane.

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- Will be computed explicitly by Riemann-Roch formula.

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- For $t > 0$, $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X}/2\right)\right]$ orbital integral for heat kernel on $C^{\infty}(X, F)$.

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- $Z\,(\gamma)$ reductive group, $\mathfrak{z}\,(\gamma) = \mathfrak{p}\,(\gamma) \oplus \mathfrak{k}\,(\gamma)$ Cartan splitting.

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Definition

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

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- Applications to eta invariants and analytic torsion.

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- The function 1 on E is L_2 and fiberwise harmonic.

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- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends to bundle of Lie algebras $TX \oplus N$.

One should expect $G \times \mathfrak{g}$ to play an important role in the construction.

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- $(\mathcal{A}(\mathfrak{g}^*), d^{\mathfrak{g}})$ resolution of $\mathbf{R}$ (algebraic Poincaré lemma).

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• Solution: tensor by $S\left(\mathfrak{g}^{*}\right)$, and use the fact that $\Lambda\left(\mathfrak{g}^{*}\right) \otimes S\left(\mathfrak{g}^{*}\right) \simeq \mathbf{R}$.

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- $C^\infty(G, \mathbf{R}) \otimes S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*) \subset C^\infty(G \times \mathfrak{g}, \mathbf{R}) \otimes \Lambda(\mathfrak{g}^*)$.

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- $S(T^*X \oplus N^*) \otimes \Lambda(T^*X \oplus N^*)$ infinite dimensional vector bundle on X.

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- $\ldots$ have representation in terms of operators acting on L_2, $\frac{\partial}{\partial Y^i} \to \frac{\partial}{\partial Y^i}, Y^j \to -\frac{\partial}{\partial Y^j} + Y^j$.
- Bargmann isomorphism, $(A(\mathfrak{g}^*), d^{\mathfrak{g}}) \to (\Omega^\bullet(\mathfrak{g}, \mathbf{R}), d^{\mathfrak{g}})$ L_2 de Rham complex with volume $\exp\left(-|Y|^2\right) dY$.

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- The quadratic term is related to the quotienting by K.

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Remark

Using the fiberwise Bargmann isomorphism, $\mathcal{L}_b^X$ acts on

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$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

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Remark

$\mathcal{L}_b^X$ not self-adjoint, not elliptic, hypoelliptic (has heat kernel).

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- Gives explicit formula for orbital integrals.

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- Welcome to Hodge theory with mass!

Langevin (C.R. de l'Académie des Sciences 1908)

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Une particule comme celle que nous considérons, grande par rapport à la distance moyenne des molécules du liquide, et se mouvant par rapport à celui-ci avec la vitesse ξ subit une résistance visqueuse égale à $-6\pi\mu a\xi$ d’après la formule de Stokes. En réalité, cette valeur n’est qu’une moyenne, et en raison de l’irrégularité des chocs des molécules environnantes, l’action du fluide sur la particule oscille autour de la valeur précédente, de sorte que l’équation du mouvement est, dans la direction x,

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X. \tag{3}$$

Sur la force complémentaire X nous savons qu’elle est indifféremment positive et négative, et sa grandeur est telle qu’elle maintient l’agitation de la particule que, sans elle, la résistance visqueuse finirait par arrêter.

P. Langevin, *Sur la théorie du mouvement brownien*, C. R. Acad. Sci. Paris **146** (1908), 530–533.

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Merci!