Density of rational points on certain del Pezzo surfaces of degree 1

jt with R. Winter

Québec-Maine number theory conference

Our objects: algebraic surfaces S over k

$S(k)$ = k-rational points of S

Density:

topology	closed set
over $\mathbb{R}$	with respect to $d(x,y)=\lvert x-y\rvert$
Zariski	algebraic subsets of S

Del Pezzo surface of degree 1

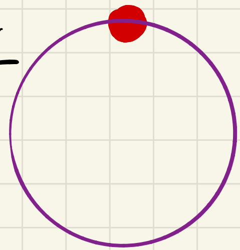
- smooth, projective, geometrically integral$/k$
- with ample $-K_X$
- $(K_X . K_X) = 1$

Elliptic surface with base $\mathbb{P}^1_k$

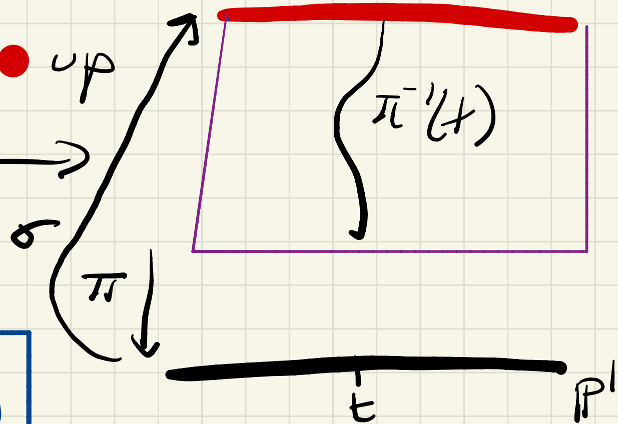
- smooth projective
- fibered in elliptic curve
  * $\pi : \mathcal{E} \longrightarrow \mathbb{P}^1_k$ is such that $\pi^{-1}(t)$ has genus 1 (finitely many exceptions)
  * there exists a section to π

$$X \longrightarrow \mathcal{E}$$

base point of $|-K_X|$



blow ● up



$\boxed{\text{Kollár: given in } \mathbb{P}(1,1,2,3) \text{ by sextic}}$

$$X: w^2 = z^3 + \underbrace{F(x,y)}_{\deg 4}z + \underbrace{G(x,y)}_{\deg 6}$$

$$\mathcal{E}: y^2 = x^3 + f(t)x + g(t)$$

sing fibers: I_1, II

● $= [0,1,0,0]$

- prove first for # field
- use argument of Colliot Thélène

Theorem (D. - Winter)

Let k be a field of characteristic 0, and let S be the del Pezzo surface of degree 1 given by the equation

$$w^2 = z^3 + Am^6 + Bn^6,$$

in $\mathbb{P}_k(1,1,2,3)$, with $A, B \in \mathbb{Z}$ non-zero. Let $\mathscr{E}$ be the elliptic surface obtained by blowing up the base point of $|-K_S|$.
Then the set of k-rational points on S is Zariski-dense in S if $\mathscr{E}$ contains a rational point which lies on a smooth fiber and is non-torsion. If k is a number field, the converse is true as well.

Why is this interesting?

$\boxed{\begin{array}{l} \textbf{Q (Manin 75): } \mathcal{E} \text{ rational e.s.} \\ \text{Is } \mathcal{E}(k) \text{ Zariski dense?} \end{array}}$

A: Yes, when a minimal model is :

$k = \#$ field

- a $dP \geq 3$ (Manin-Segre 75)
- a conic bundle (Kollár-Mella 14)
- A $dP2$ with a point outside of finite set (intersections of 4 exc. curve, ramification locus of $-K_X$)

(Várilly-Alvarado, Testa, Salgado 14)

Arithmetic strategy: root number $k = \mathbb{Q}$

$$W = (-1)^{\text{rkan}} \left(\overset{\text{under BSD}}{= (-1)^{\text{rk}}} \right)$$

Conditional Answer:

- $\mathcal{E}$ is non-isotrivial (D.16)

- $\mathcal{E}$ is isotrivial and

  $j \neq 0$ (D.16)

  $j = 0$ and not of the form (Varilly-Alvarado ″)

$$y^2 = x^3 + \alpha A(t)^2 + \beta B(t)^2$$

in particular not

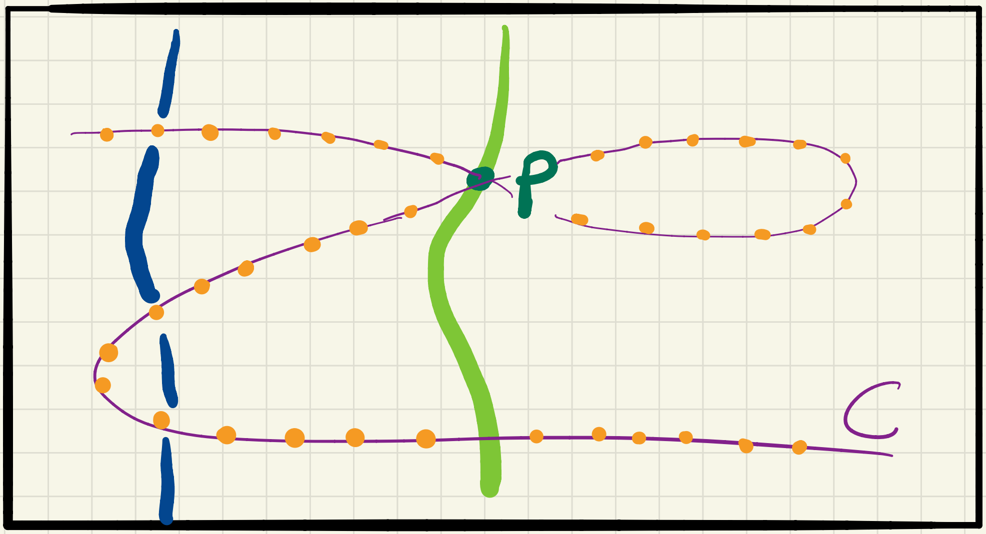
$$y^2 = x^3 + At^6 + B$$

Important fact:

$$\boxed{\begin{gathered}\#\left\{ t \in \mathbb{P}^1 \,/\, \pi^{-1}(t) \text{ non-zero rank} \right\} = \infty \\ \Updownarrow \\ \text{Zariski-density}\end{gathered}}$$

Our strategy: construct a multisection with a lot of points.

Sketch of proof



Let P be non-torsion on its fiber.

① There exists a 3-section C passing twice through P.

② The normalisation $\tilde{C}$ is an elliptic curve with positive rank

③ Each point of C is non-torsion on its fiber.

② $\sigma : \mathbb{P}(1,1,2,3) \longrightarrow \mathbb{P}(1,1,2,3)$

$(m, n, z, w) \longmapsto (\zeta_3^2 x : y : \zeta_3 z : w)$

autom. of C