Double character sum and double Dirichlet series

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Double character sum

We will study the double character sum

$$S(X,Y) = \sum_{\substack{m \le X, \\ m \text{ odd}}} \sum_{\substack{n \le Y, \\ n \text{ odd}}} \chi_m(n),$$

where $\chi_m(n) = \left(\frac{m}{n}\right)$.

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If $m=\square$:

$$\sum_{n\le Y,\ n \text{ odd}}\chi_m(n)=\sum_{\substack{n\le Y,\ n \text{ odd},\\ (m,n)=1}}1=\frac{Y}{2}\frac{\varphi(m)}{m}+O(Y^{\varepsilon}).$$

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If $m \ne \square$, Pólya-Vinogradov gives

$$\sum_{n \le Y} \chi_m(n) \ll m^{1/2} \log m.$$

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Hence

$$S(X,Y) = \left(\frac{Y}{2} + O(Y^{\varepsilon})\right) \sum_{\substack{m \le \sqrt{X}, \\ m \text{ odd}}} \frac{\varphi(m)}{m} + O\left(\sum_{m \le X} m^{1/2} \log m\right) =$$

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What if X, Y have similar size?

Double character sum

Theorem (Conrey, Farmer, Soundararajan (2000))

Uniformly for all large X, Y, we have:

$$S(X,Y) = \frac{2}{\pi^2} C\left(\frac{Y}{X}\right) X^{3/2} + O\left(\left(XY^{7/16} + YX^{7/16}\right)\log XY\right),$$

where

$$C(\alpha) = \sqrt{\alpha} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\alpha}\sqrt{y}\left(1 - \cos\left(\frac{2\pi k^2}{y}\right) + \sin\left(\frac{2\pi k^2}{y}\right)\right) dy.$$

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The proof is by Poisson summation and estimation of the sums of Gauss sums.

Double character sum

From now on, we will work with the smooth sum

$$S(X,Y)=\sum_{m \text{ odd}}\sum_{n \text{ odd}} \chi_m(n)\varphi(m/X)\psi(n/Y),$$

where φ, ψ are smooth weights supported in $(0,1)$.

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The inverse Mellin formula says

$$\varphi(x) = \frac{1}{2\pi i}\int_{(c)} \hat{\varphi}(s)x^{-s}ds,$$

where the integral is over the vertical line $\mathrm{Re}(s) = c$.

If φ is smooth, then $\hat{\varphi}(\sigma + it) \ll_{A,\sigma} (1 + |t|)^{-A}$ for any A.

Main result

Theorem (Č., 2020+)

Let $\varepsilon > 0$. Then for all large X, Y, we have

$$S(X,Y) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X}; \varphi, \psi\right) + O_\varepsilon(XY^\delta + YX^\delta),$$

where $\delta = \varepsilon$, and

$$D(\alpha;\varphi,\psi) = \frac{\hat{\varphi}(1)\hat{\psi}\left(\frac{1}{2}\right)\alpha^{1/2} + \hat{\psi}(1)\hat{\varphi}\left(\frac{1}{2}\right)\alpha}{2} +$$

$$+ \frac{1}{i\sqrt{\pi}} \int_{(3/4)} \left(\frac{\alpha}{2\pi}\right)^s \cdot \hat{\varphi}\left(\frac{3}{2} - s\right)\hat{\psi}(s)\Gamma\left(s - \frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)ds.$$

If we assume the Riemann Hypothesis, then we can take $\delta = -1/4 + \varepsilon$.

For $\varphi = \psi = 1_{[0,1]}$, one can show that $D(\alpha;\varphi,\psi) = C(\alpha)$.

Proof idea

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$$= \left(\frac{1}{2\pi i}\right)^2 \int_{(\sigma)} \int_{(\omega)} A(s,w) X^w Y^s \hat{\varphi}(w) \hat{\psi}(s) dw ds,$$

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where the subscript 2 means the Euler factor at 2 is removed, and

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A(s,w) = \sum_{m \text{ odd}} \frac{L_2(s,\chi_m)}{m^w} = \sum_{m \text{ odd}} \sum_{n \text{ odd}} \frac{\chi_m(n)}{m^w n^s}
$$

is a double Dirichlet series, absolutely convergent if $\mathrm{Re}(s), \mathrm{Re}(w)$ are large enough.

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- Polar lines $s=1$, $w=1$, $s+w=3/2$ (and others)

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We expect two functional equations:

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To make this heuristic rigorous – need to deal with non-primitive characters, exact quadratic reciprocity and functional equations.
Blomer worked out the details in this case. He showed that

$$Z(s,w) := A(s,w)\zeta_2(2s+2w-1)$$

has meromorphic continuation with polar lines $s=1$, $w=1$ and $s+w=3/2$, and satisfies some functional equations.
Part of a general theory of Bump, Diaconu, Friedberg, Goldfeld, Hoffstein, and others

Meromorphic continuation (due to Blomer)

Consider

$$Z(s,w;\psi,\psi') := \zeta_2(2s+2w-1) \sum_{m \text{ odd}} \frac{L_2\left(s,\chi_m\psi\right)\psi'(m)}{m^w},$$

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Meromorphic continuation (due to Blomer)

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$$= \sum_{d \text{ odd}} \frac{\mu(d)\chi_{m_0}(d)\psi(d)}{d^{s+2w}} \sum_{m_1 \text{ odd}} \frac{1}{m_1^{2w}} = \frac{\zeta_2(2w)}{L_2(s+2w, \chi_{m_0}\psi)},$$

so

$$Z(s, w; \psi, \psi') = \zeta_2(2s + 2w - 1) \sum_{\substack{m_0 \text{ odd}, \\ \mu^2(m_0)=1}} \frac{L_2(s, \chi_{m_0}\psi)\psi'(m_0)\zeta_2(2w)}{m_0^w L_2(s+2w, \chi_{m_0}\psi)}.$$

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Notice that $(s, w) \mapsto (1 - s, s + w - 1/2)$ interchanges $2s + 2w - 1$ with $2w$ and preserves $s + 2w$.

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The sum converges absolutely for $\mathrm{Re}(w) > 1$ and $\mathrm{Re}(s + w) > 3/2$ unless ψ is the trivial character, in which case the first summand has a pole at $s = 1$ with residue $\zeta_2(2w)/2$.

Meromorphic continuation (due to Blomer)

$$Z(s,w;\psi,\psi') = \zeta_2(2s+2w-1) \sum_{\substack{m_0 \text{ odd}, \\ \mu^2(m_0)=1}} \frac{L_2(s,\chi_{m_0}\psi)\psi'(m_0)\zeta_2(2w)}{m_0^w L_2(s+2w,\chi_{m_0}\psi)}.$$

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Using the functional equation for $L(s,\chi_{m_0}\psi)$, we can find a 16×16 matrix $B(s)$ such that

$$\mathrm{Z}(s,w) = B(s)\mathrm{Z}(1-s,s+w-1/2)$$

where

$$\mathrm{Z}(s,w,\psi) = \begin{pmatrix} Z(s,w;\psi,\psi_1) \\ Z(s,w;\psi,\psi_{-1}) \\ Z(s,w;\psi,\psi_2) \\ Z(s,w;\psi,\psi_{-2}) \end{pmatrix}, \quad \mathrm{Z}(s,w) = \begin{pmatrix} \mathrm{Z}(s,w,\psi_1) \\ \mathrm{Z}(s,w,\psi_{-1}) \\ \mathrm{Z}(s,w,\psi_2) \\ \mathrm{Z}(s,w,\psi_{-2}) \end{pmatrix},$$

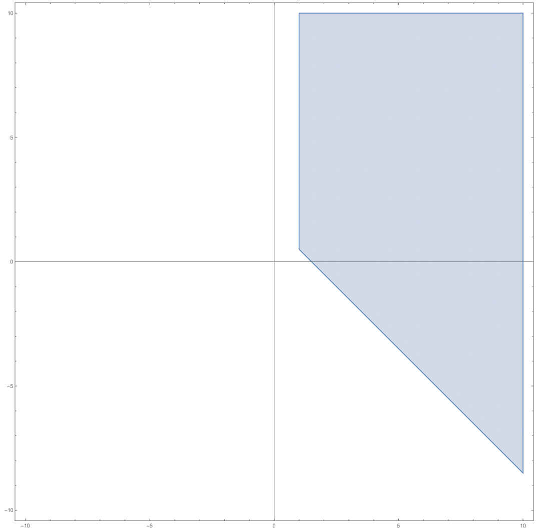
and $\psi_j(n) = \left(\frac{j}{n}\right)$.

Meromorphic continuation (due to Blomer)

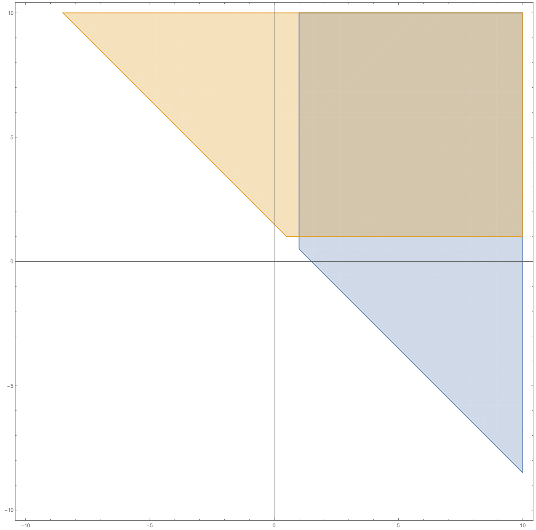
The use of quadratic reciprocity gives a 16×16 matrix A such that

$$Z(s, w) = A \cdot Z(w, s)$$

Meromorphic continuation



Meromorphic continuation



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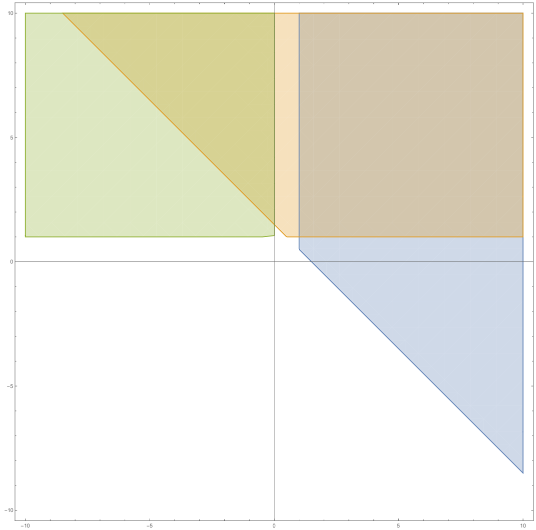
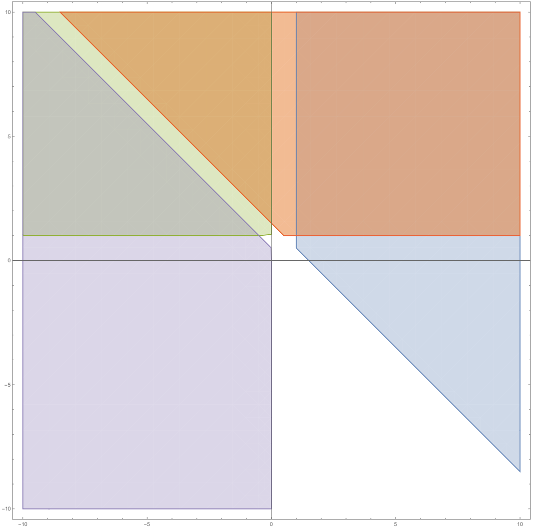
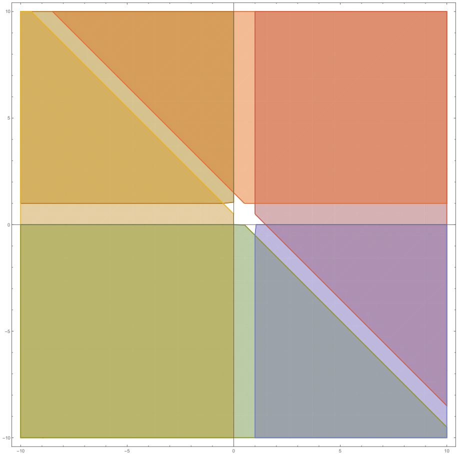


Meromorphic continuation



Meromorphic continuation



We get meromorphic continuation with polar lines $s = 1$, $w = 1$, $s + w = 3/2$ (other cancelled by gamma factors) outside a "compact domain". Use Bochner's Tube Theorem on the holomorphic $(s-1)(w-1)(s+w-3/2)Z(s, w; \psi, \psi')$ to continue everywhere. $A(s, w)$ is polynomially bounded in vertical strips.

Poles of $A(s, w)$

$A(s,w) = \frac{Z(s,w;\psi_1,\psi_1)}{\zeta_2(2s+2w-1)}$ has the following polar lines:

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$$\mathrm{Res}_{\,(s,\frac{3}{2}-s)} A(s,w) = \frac{\sqrt{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{2\zeta_2(2)(2\pi)^s},$$

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  - the lines $s + w = \frac{k\pi i}{\log 2} + \frac{1}{2}$, which have $\mathrm{Re}(s+w) = \frac{1}{2}$.

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Shifting the integral

$$S(X, Y) = \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \int_{(2)} A(s, w) X^w Y^s \hat{\varphi}(w) \hat{\psi}(s) dw ds =$$

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$$R = X\hat{\varphi}(1) \frac{1}{2\pi i} \int\limits_{(2)} \frac{Y^s \zeta_2(2s)}{2\zeta_2(2s+1)} \hat{\psi}(s) ds.$$

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The last integral has a pole at $s = 1/2$ with residue $\frac{Y^{1/2}\hat{\psi}(1/2)}{8\zeta_2(2)}$, and at $s = \frac{\rho-1}{2}$, which have $\mathrm{Re}(s) \leq c$, where $c = 0$ or $c = -1/4$ under RH.

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The last integral has a pole at $s = 1/2$ with residue $\frac{Y^{1/2}\hat{\psi}(1/2)}{8\zeta_2(2)}$, and at $s = \frac{\rho-1}{2}$, which have $\mathrm{Re}(s) \leq c$, where $c = 0$ or $c = -1/4$ under RH. Hence for $\delta = c + \varepsilon$,

$$R = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2)}{8\zeta_2(2)} + O(XY^\delta).$$

Shifting the integral

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2)}{8\zeta_2(2)} + \\ + \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \int_{\left(\frac{3}{4}+\varepsilon\right)} A(s,w) X^w Y^s \hat{\varphi}(w)\hat{\psi}(s) dw ds + O(XY^{\delta}).$$

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If $\varphi = \psi = 1_{[0,1]}$, then the first term is $\frac{2XY^{1/2}}{\pi^2}$, which corresponds to the contribution when $m = \square$.

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The contribution of the polar line $s = 1$ is completely analogous, so

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{8\zeta_2(2)} + \\ + \left(\frac{1}{2\pi i}\right)^2 \int_{(3/4)} \int_{(3/4+\varepsilon)} A(s,w) X^w Y^s \hat{\varphi}(w)\hat{\psi}(s) dwds + O(XY^\delta + YX^\delta).$$

Shifting the integral further

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{8\zeta_2(2)} +$$

$$+\left(\frac{1}{2\pi i}\right)^2 \int_{(3/4)} \int_{(3/4+\varepsilon)} A(s,w) X^w Y^s \hat{\varphi}(w)\hat{\psi}(s)\, dw ds + O(XY^\delta + YX^\delta)$$

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$$= \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{8\zeta_2(2)} + Q$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \int_{(3/4)} \int_{(\delta')} A(s,w) X^w Y^s \hat{\varphi}(w)\hat{\psi}(s) dwds + O(XY^\delta + YX^\delta),$$

Shifting the integral further

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{8\zeta_2(2)} +$$
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where

$$Q = \frac{1}{2\pi i} \int_{(3/4)} X^{3/2-s} Y^s \hat{\varphi}(3/2-s)\hat{\psi}(s) \frac{\sqrt{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{2\zeta_2(2)(2\pi)^s} ds.$$

Shifting the integral further

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{8\zeta_2(2)} +$$

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We can take $\delta' = 1/4 + \delta$, so the last double integral is

$$\ll X^{1/4+\delta} Y^{3/4} \ll XY^\delta + YX^\delta.$$

Final result

Putting everything together gives our final result:

$$S(X,Y) = \frac{XY^{1/2}\hat{\varphi}(1)\hat{\psi}(1/2) + YX^{1/2}\hat{\varphi}(1/2)\hat{\psi}(1)}{\pi^2} +$$

$$+ \frac{2X^{3/2}}{\pi^2} \cdot \frac{1}{\sqrt{\pi} i} \int_{(3/4)} \left(\frac{Y}{2\pi X}\right)^s \hat{\varphi}\left(\frac{3}{2} - s\right) \hat{\psi}(s) \sin\left(\frac{\pi s}{2}\right) \Gamma\left(s - \frac{1}{2}\right) \zeta(2s-1) ds +$$

$$+ O(XY^\delta + YX^\delta) =$$

$$= \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X}; \varphi, \psi\right) + O(XY^\delta + YX^\delta).$$

Thank you.