GEMSTONE minicourse

The Steklov problem on non-smooth domains

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The bibliography will be mentioned as we proceed and the list with references given in the end, but also

advertisement

most of the material of these lectures + much more will be in the forthcoming book ML + Dan Mangoubi + Iosif Polterovich, *Topics in Spectral Geometry*, a draft copy of which will be available from author's websites soon

Plan:

Today: Basics

Method of multipliers

Wednesday: Applications

Bounds for eigenvalues of DtN map

Friday: Steklov problem in polygons

Notational / terminology conventions

✓ $\mathbb{N} = \{1, 2, \dots\}$

In $\mathbb{R}^d$, $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$

no minus!

In $\mathbb{R}^d$, $\langle \vec{u}, \vec{v} \rangle = \sum_{j=1}^{d} u_j v_j$

all vectors are column vectors

In $\mathbb{C}^d$, $\langle \vec{u}, \vec{v} \rangle = \sum_{j=1}^{d} u_j \overline{v_j}$

$|\vec{u}| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

In an abstract Hilbert space $\mathcal{H}$,

the inner product is $(u, v)_{\mathcal{H}}$ and

the norm is $\|u\|_{\mathcal{H}}$

In $\mathbb{R}^d$,

$$\text{grad}\, u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right) \quad \text{gradient}$$

$$\text{div}\, \vec{u} = \frac{\partial u_1}{\partial x_1} + \dots + \frac{\partial u_d}{\partial x_d} \quad \text{divergence}$$

and therefore $\Delta = \text{div}\, \nabla$

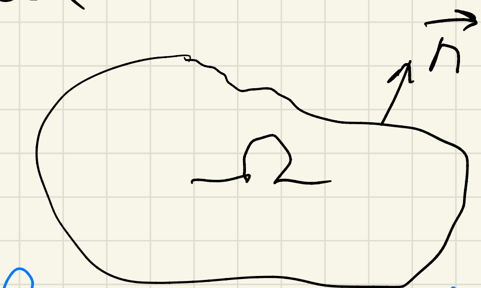
$$\mathbb{B}^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$$

$$S^{d-1} = \partial \mathbb{B}^d = \{x \in \mathbb{R}^d, |x| = 1\}$$

$\Omega \subset \mathbb{R}^d$ usually a domain (bdd open connected set). Connectedness not required but assumed for simplicity

with boundary $\partial\Omega$ often $\boxed{\partial\Omega =: M}$

$\vec{n}$ - exterior unit normal



Ω Lipschitz $\iff$ Ω has Lipschitz bdry

$\iff$ $\partial\Omega$ can be locally represented as a graph of a Lipschitz function

PART 0.

DEFINITIONS & BASICS

MAIN objects: various spectral problems involving Laplacian

Ω – a domain in $\mathbb{R}^d$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

spectral problem for the Dirichlet Laplacian

We are looking for $\lambda \in \mathbb{R}$ (eigenvalues) s.t. there exists $u \neq 0$ (eigenfunction) solving this problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \partial_n u \equiv (n \cdot \nabla u) = 0 & \text{on } \partial\Omega \end{cases}$$

assumed Lipschitz

spectral problem for the Neumann Laplacian

for a given $\gamma \in \mathbb{R}$:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \partial_n u + \gamma u = 0 & \text{on } \partial\Omega \end{cases}$$

spectral problem for the Robin Laplacian

For either of the Dirichlet, Neumann or Robin cases, we have

note: $-\Delta^{R,0} = -\Delta^{N}$

- we can construct unbounded self-adjoint operators $-\Delta^D_\Omega$, $-\Delta^N_\Omega$, $-\Delta^{R,\gamma}_\Omega$ acting in $L^2(\Omega)$, describe their domains, and treat the spectral problems in the operator-theoretical sense
- in each case, the spectrum $\mathrm{Spec}(-\Delta^\chi)$, $\chi \in \{D; N; R,\gamma\}$ is discrete and consists of eigenvalues of finite multiplicity accumulating only to $+\infty$.

- We will denote the eigenvalues by

$$(0<)\,\lambda_1^D < \lambda_2^D \leq \dots \leq \lambda_m^D \leq \dots \qquad \text{Dir}$$

$$0 = \lambda_1^N < \lambda_2^N \leq \dots \qquad \text{Neu}$$

$$\lambda_1^{R,\gamma} \leq \lambda_2^{R,\gamma} \leq \dots \qquad \text{Robin}$$

- The corresponding eigenfunctions $\{u_m^{\alpha}\}_{m=1}^{\infty}$ in each case form a basis in $L^2(\Omega)$, which may be chosen to be orthogonal.

- We have the variational principles

$$\lambda_k^D = \min_{\substack{\mathcal{L} \subset H_0^1(\Omega) \\ \dim \mathcal{L} = k}} \max_{u \in \mathcal{L} \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \quad // \quad \frac{(-\Delta u, u)}{\|u\|^2}$$

$$\lambda_k^N = \min_{\substack{\mathcal{L} \subset H^1(\Omega) \\ \dim \mathcal{L} = k}} \max_{u \in \mathcal{L} \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}$$

$$\lambda_k^{R,\gamma} = \min_{\substack{\mathcal{L} \subset H^1(\Omega) \\ \dim \mathcal{L} = k}} \max_{u \in \mathcal{L} \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)} + \gamma \|u\|^2_{L^2(\partial\Omega)}}{\|u\|^2_{L^2(\Omega)}}$$

- As follows from variational principles, for $\gamma_2 \geq \gamma_1$ we have

$$\lambda_k^{R,\gamma_1} \leq \lambda_k^{R,\gamma_2} \leq \lambda_k^{D} \qquad k \in \mathbb{N}$$

and

$$\lambda_k^{N} \leq \lambda_k^{D}$$

(this can be significantly strengthened!)

- Let us define, in each case, the eigenvalue counting function

$$\mathcal{N}^{X}(\lambda) := \#\{j : \lambda_j^{X} \leq \lambda\}$$

Then we have Weyl's laws:

$$\mathcal{N}^{\lambda}(\lambda) = C_d \, |\Omega|_d \, \lambda^{d/2} + o(\lambda^{d/2}),$$
as $\lambda \to +\infty$

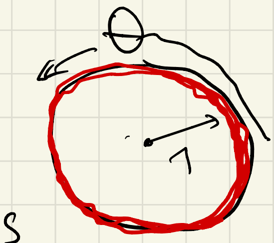
$$C_d = \frac{|\mathbb{B}^d|_d}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2} \, \Gamma(1 + d/2)}$$

We will also use the spectrum of the Laplace-Beltrami operator on a smooth compact closed Riemannian manifold (M, g) of dimension d, $g = \{g_{ij}\}_{i,j=1}^{d}$,

$$-\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\,\frac{\partial f}{\partial x_j}\right) = \lambda f$$

The spectrum is discrete, $0 = \lambda_1(M) < \lambda_2(M) \leq \dots$, and the same variational principle as in the Neumann case applies. A simila Weyl's Law also holds.

A particular example will be important in the sequel.



__Example__. Let $M = S^1$. Then the eigenvalues and eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^1}$ are

$$\lambda_1 = 0 \qquad u_1 = 1$$

$$\lambda_{2j} = \lambda_{2j+1} = j^2 \qquad \operatorname{span}\{u_{2j}, u_{2j+1}\}$$

$$= \operatorname{span}\{\cos(j\theta), \sin(j\theta)\}$$

$$j \in \mathbb{N}$$

Steklov problem:

For the moment, assume that $\Omega \subset \mathbb{R}^d$ is a bdd domain with smooth boundary $\partial\Omega$. Let $u \in C^\infty(\partial\Omega)$. We define its harmonic extension $U := \mathcal{E}_0 u$ as the unique soln of the non-homogeneous Dirichlet problem

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U|_\Gamma = u \end{cases}$$

The Dirichlet-to-Neumann (DtN) map is the operator $\mathcal{D}_0 : u \longmapsto (\partial_n \mathcal{E}_0 u)|_{\partial\Omega}$

$$\| \quad$$

$$(\partial_n U)|_{\partial\Omega}$$

The Steklov problem is the eigenvalue problem for the operator $\mathcal{D}_0$: find $\sigma \in \mathbb{R}$ and $U \neq 0$ s.t.

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ \partial_n U = \sigma U & \text{on } \partial\Omega \end{cases}$$

the spectral parameter is in the boundary condition!

Note a terminological dichotomy: if

$$\Delta U = 0, \quad \partial_n U = \sigma U,$$

we call U an eigenfunction of the Steklov problem.

At the same time, let $u = U|_{\partial\Omega}$, so that $U = \mathcal{E}_0 u$, and then $\mathcal{D}_0 u = \sigma u$, and we will call u an eigenfunction of the DtN map $\mathcal{D}_0$.

<u>Notation</u> $\mathcal{H}_0(\Omega) = \{U \in H^1(\Omega) ; \Delta U = 0\}$

the space of harmonic functions

The definition of $\mathcal{D}_0 u$ can be extended to the case when $\partial\Omega$ is Lipschitz, and also when $u \in H^{1/2}(\partial\Omega)$, in which case $\mathcal{E}_0 u \in \mathcal{H}_0(\Omega)$, and $\mathcal{D}_0 u \in H^{-1/2}(\partial\Omega)$.
It can be shown that the operator $\mathcal{D}_0$ defined in this way is a self-adjoint operator in $L^2(\partial\Omega)$ with a discrete spectrum $0 = \sigma_1(\Omega) < \sigma_2(\Omega) \leq \dots \nearrow +\infty$

Moreover, it can be shown that the eigenfunctions of the DtN map form a basis in $L^2(\partial\Omega)$ and can be chosen to be orthogonal there

Important example Let $\Omega = \mathbb{D} \subset \mathbb{R}^2$ be the unit disk. Then, by separation of variables, its Steklov eigenvalues and eigenfunctions are

$$0, 1, 1, \dots, k, k, \dots \qquad k \in \mathbb{N}$$

$$1, r\sin\theta, r\cos\theta, \dots, r^k \sin k\theta, r^k \cos k\theta, \dots$$

The Steklov eigenvalues of the disk are <u>exactly</u> the square roots of eigenvalues of $-\Delta_{S^1}$.

The variational principle for $\mathcal{D}_0$.

Using integration by parts, we immediately see

$$(\mathcal{D}_0 u, u)_{L^2(\partial\Omega)} = \int_{\partial\Omega} \frac{\partial U}{\partial n} U = \int_{\Omega} |\nabla U|^2 \checkmark$$

$$\boxed{U = \mathcal{E}_0 u} \qquad = \|\nabla \mathcal{E}_0 u\|^2_{L^2(\Omega)}$$

The classical variational principle implies

$$\sigma_k = \min_{\substack{\mathcal{L} \subset H^{1/2}(\partial\Omega) \\ \dim \mathcal{L} = k}} \max_{u \in \mathcal{L}\setminus\{0\}} \frac{\|\nabla \mathcal{E}_0 u\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\partial\Omega)}} .$$

This can be rewritten as

$$(*) \quad \sigma_k = \underbrace{\min_{\mathcal{L} \subset \mathcal{H}_0(\Omega)}}_{\dim \mathcal{L} = k} \max_{U \in \mathcal{L} \setminus \{0\}} \frac{\|\nabla U\|^2_{L^2(\Omega)}}{\|U|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}}$$

But taking subspaces of the space of harmonic functions is awkward. Fortunately, we have

Prop. In $(*)$, one can replace $\mathcal{H}_0(\Omega)$ with $H^1(\Omega)$

To prove this, we need

Lemma. Let Ω be a bdd open set in $\mathbb{R}^d$. Then

$$(**) \qquad \boxed{H^1(\Omega) = \mathcal{H}_0(\Omega) \oplus H^1_0(\Omega)}$$

The direct sum is not orthogonal, however

$$(***) \quad (\nabla U, \nabla V)_{L^2(\Omega)} = 0 \quad \text{for any } U \in \mathcal{H}_0(\Omega),\ V \in H^1_0(\Omega)$$

Proof. Let $W \in H^1(\Omega)$. Set $u \in W|_{\partial\Omega}$, $U = \mathcal{E}_0 u \in \mathcal{H}_0(\Omega)$.

Then $V = W - U \in H^1_0(\Omega)$ since $U|_{\partial\Omega} = 0$. As $\mathcal{H}_0(\Omega) \cap H^1_0(\Omega) = \{0\}$, $(**)$ follows.

To prove $(***)$, we integrate by parts:

$$(\nabla U, \nabla V)_\Omega = (-\underset{\overset{\shortparallel}{0}}{\Delta U}, V)_\Omega + (\partial_n U, \underset{\overset{\shortparallel}{0}}{V})_{\partial\Omega} = 0$$

Proof of the Proposition.

We take $W \in H^1(\Omega)$, represent it as $W = \underset{\mathcal{H}_0(\Omega)}{U} + \underset{H_0^1(\Omega)}{V}$, and estimate Steklov's Rayleigh quotient

$$\frac{\|\nabla W\|_\Omega^2}{\|W|_{\partial\Omega}\|_{\partial\Omega}^2} = \frac{\|\nabla(U+V)\|_\Omega^2}{\|U|_{\partial\Omega}\|_{\partial\Omega}^2} = \frac{\|\nabla U\|_\Omega^2 + \|\nabla V\|_\Omega^2}{\|U|_{\partial\Omega}\|_{\partial\Omega}^2}$$

$$\geqslant \frac{\|\nabla U\|_\Omega^2}{\|U|_{\partial\Omega}\|_{\partial\Omega}^2} + \lambda_1^D(\Omega)\,\frac{\|V\|_\Omega^2}{\|U|_{\partial\Omega}\|_{\partial\Omega}^2}\,.$$

Minimisation procedure requires taking $V=0$ ∎

General task of Spectral geometry:
find relations of spectra of various operators to underlying geometry

General difficulty:
Very few problems can be solved explicitly (or semi-explicitly, in terms of roots of some special functions or transcendental eqns): cuboids, balls, cylinders.

Hence the need for bounds/asymptotics.

Part I.

METHOD OF MULTIPLIERS

Very general idea: let H be a self-adjoint operator acting in some Hilbert space $\mathcal{H}$, semi-bounded below, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and orthonormal basis of eigenfunctions $\{u_j\}$, so that

$$H u_j = \lambda u_j$$

Let us act on both sides by some "suitable" self-adjoint operator G, then multiply by u_j in $(\, , \,)_{\mathcal{H}}$, and see what happens

a multiplier

For now, we act formally:

$$GHu_j = \lambda_j G u_j \implies HGu_j - GHu_j = (H-\lambda_j)Gu_j$$

$$\implies [H,G]u_j = (H-\lambda_j)Gu_j$$

$$(G[H,G]u_j, u_j)_{\mathcal{H}} = (G(H-\lambda_j)Gu_j, u_j)_{\mathcal{H}}$$

$$(-[[H,G],G]u_j, u_j) + ([H,G]Gu_j, u_j)$$
$$= -([[H,G],G]u_j, u_j)$$
$$- (u_j, G[H,G]u_j)$$
$$= -\frac{1}{2}([[H,G],G]u_j, u_j)$$

$$((H-\lambda_j)Gu_j, Gu_j)$$
$$= \sum_{k=1}^{\infty} ((H-\lambda_j)Gu_j, u_k)(u_k, Gu_j)$$
$$= \sum_{k=1}^{\infty} (\lambda_k - \lambda_j) |(Gu_j, u_k)|^2$$

To make this rigorous, we need to make sure all operations are well defined

Thm [ML + Parnovski '02, following Harrell-Stubbe '97]

Let $H = H^*$ be as above, and let $G = G^*$ s.t. $G(\mathrm{Dom}(H)) \subseteq \mathrm{Dom}(H) \subseteq \mathrm{Dom}(G)$. Then for each fixed $j \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \frac{|([H,G]u_j, u_k)_{\mathcal{H}}|^2}{\lambda_k - \lambda_j} = \sum_{k=1}^{\infty} (\lambda_k - \lambda_j) |(Gu_j, u_k)_{\mathcal{H}}|^2$$

$$= -\frac{1}{2} ([[H,G],G]u_j, u_j)_{\mathcal{H}}$$

$\boxed{\frac{0}{0} := 0}$

an abstract commutator trace identity

Proof: we already proved the second equality + we use $([H,G]u_j, u_k) = (\lambda_k - \lambda_j)(Gu_j, u_k)$ ■

Example. Let $H = -\Delta^D_\Omega$, $\Omega \in \mathbb{R}^d$.

$$G = x_\ell \cdot \quad , \quad \ell = 1, \dots, d$$

$$[H, G]u = -\Delta(x_\ell u) + x_\ell \Delta u = -2\langle \nabla x_\ell, \nabla u\rangle$$

$$= -2\frac{\partial u}{\partial x_\ell}$$

$$[[H,G],G]u = -2\frac{\partial(x_\ell u)}{\partial x_\ell} + 2x_\ell \frac{\partial u}{\partial x_\ell}$$

$$= -2u \text{, and we get}$$

$$\boxed{4\sum_{k=1}^{\infty} \frac{\left(\int_\Omega \frac{\partial u_j}{\partial x_\ell} u_k\right)^2}{\lambda_k - \lambda_j} = \sum_{k=1}^{\infty} (\lambda_k - \lambda_j)\left(\int_\Omega x_\ell u_j u_k\right)^2 = 1}$$

known in quantum mechanics since 1920s!

Let us play a bit more, in case of $-\Delta^D_\Omega$, with

$$\frac{1}{4} = \sum_{k=1}^{\infty} \frac{|(\partial u_j/\partial x_\ell, u_k)_\Omega|^2}{\lambda_k - \lambda_j}$$

First, sum over $j = 1, \dots, m$ for a fixed $m \in \mathbb{N}$, and then estimate; using shorthand $u_{j,\ell} = \frac{\partial u_j}{\partial x_\ell}$

$$\frac{m}{4} = \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{|(u_{j,\ell}, u_k)|^2}{\lambda_k - \lambda_j} = \sum_{j=1}^{m} \sum_{k=m+1}^{\infty} \frac{|(u_{j,\ell}, u_k)|^2}{\lambda_k - \lambda_j}$$

all the terms with $k \leq m$ cancel out

$$\leq \frac{1}{\lambda_{m+1} - \lambda_m} \sum_{j=1}^{m} \sum_{k=1}^{\infty} |(u_{j,\ell}, u_k)|^2 = \sum_{j=1}^{m} \left\| \frac{\partial u_j}{\partial x_\ell} \right\|^2_{L^2(\Omega)}$$

smallest denominator and extend the sum

Thus, we have

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{m} \sum_{j=1}^{m} \left\| \frac{\partial u_j}{\partial x_\ell} \right\|^2_{L^2(\Omega)}$$

Now sum over $\ell = 1, \dots, d$, divide by d, and use $\|\nabla u_j\|^2 = \lambda_j$. We arrive at

$$\lambda^D_{m+1} - \lambda^D_m \leq \frac{4}{dm} \sum_{j=1}^{m} \lambda^D_j \qquad m \in \mathbb{N}$$

Payne - Pólya - Weinberger (1956)
universal inequality for the Dirichlet Laplacian

So, we have got some inequalities with almost no work. Further (better) bounds can be derived in a general setting.

Advantages and disadvantages of commutator techniques

+	−
• "soft" analysis • general scheme	• hard to apply to Neumann/Robin — difficult to find G to preserve $\mathrm{Dom}(H)$ • hard to apply to DtN map — how does one compute commutators?

We therefore use a different flavour of the method of multipliers, which is variably associated with Franz Rellich (1940s), Lars Hörmander (1950s), Stanislav Pohozhaev (1960s), and Cathleen Morawetz (1970s).

For the moment, we work only with harmonic functions: $\Delta U = 0$ in $\Omega \subset \mathbb{R}^d$

Basic idea: use the multiplier $\langle \vec{F}, \nabla \rangle$, where $\vec{F}(x)$ is a suitable vector field

We will need the following notions and identities from vector calculus:

$\Omega \subset \mathbb{R}^d$ — an open set

$a : \Omega \to \mathbb{R}$, $\vec{A}, \vec{B} : \Omega \to \mathbb{R}^d$ — sufficiently smooth

$\mathrm{Jac}_A := \left(\dfrac{\partial A_i}{\partial x_j} \right)_{i,j=1}^d$ the Jacobian of $\vec{A}$

$\mathrm{Hes}_a := \mathrm{Jac}_{\nabla a} = \left(\dfrac{\partial^2 a}{\partial x_i \partial x_j} \right)_{i,j=1}^d$ the Hessian of a

for any linear operator (matrix) $C : \mathbb{R}^d \to \mathbb{R}^d$ we write $C[\vec{A}, \vec{B}] := \langle C\vec{A}, \vec{B} \rangle$ and denote by C^* the transpose of C

The following identities are standard (exercise):

(1) $\operatorname{div}(a\vec{A}) = \langle \nabla a, \vec{A} \rangle + a \operatorname{div} \vec{A}$

(2) $\nabla \langle \vec{A}, \vec{B} \rangle = \mathrm{Jac}^{*}_{\vec{A}}\, \vec{B} + \mathrm{Jac}^{*}_{\vec{B}}\, \vec{A}$

(3) $\nabla \left(|\nabla a|^{2} \right) = 2\, \mathrm{Hes}_{a}\, \nabla a$

We will now prove the following

Thm. (Generalised Pohozhaev's identity for harmonic functions [Colbois, Girouard, Hassannezhad, 2018])

Let $\Omega \subset \mathbb{R}^d$ be a bdd domain with smooth boundary $M = \partial\Omega$. Let $\vec{F}$ be a smooth vector field on $\overline{\Omega}$, let $u \in H^1(M)$, and $U = \mathcal{E}_0 u$ a unique harmonic extension of u onto Ω. Then

$$\int_M \langle \vec{F}, \nabla U \rangle \partial_n U - \frac{1}{2}\int_M |\nabla U|^2 \langle \vec{F}, \vec{n} \rangle$$

$$+ \frac{1}{2}\int_\Omega |\nabla U|^2 \operatorname{div} \vec{F} - \int_\Omega \operatorname{Jac}_{\vec{F}}[\nabla U, \nabla U] = 0.$$

Proof: We have $\Delta U = \operatorname{div} \nabla U = 0$ in Ω.

Using vector identities (1) and (2), we get

$$\operatorname{div}\left(\langle \vec{F}, \nabla U\rangle \nabla U\right) = \langle \nabla \langle \vec{F}, \nabla U\rangle, \nabla U\rangle$$

$$= \operatorname{Jac}_{\vec{F}}[\nabla U, \nabla U] + \operatorname{Hes}_U[\vec{F}, \nabla U]$$

Using (1) and (3), we get

$$1/2 \operatorname{div}\left(|\nabla U|^2 \vec{F}\right) = \operatorname{Hes}_U[\vec{F}, \nabla U] + \frac{1}{2}|\nabla U|^2 \operatorname{div} \vec{F}$$

Subtracting, we get

$$\operatorname{div}\left(\langle \vec{F}, \nabla U\rangle \nabla U - \frac{1}{2}|\nabla U|^2 \vec{F}\right) =$$

$$\operatorname{Jac}_{\vec{F}}[\nabla U, \nabla U] - \frac{1}{2}|\nabla U|^2 \operatorname{div} \vec{F}$$

Finally, we integrate over Ω and use the Divergence Thm. $\blacksquare$

Let me repeat the result:

$$\int_M \langle \vec{F}, \nabla U \rangle \partial_n U - \frac{1}{2}\int_M |\nabla U|^2 \langle \vec{F}, \vec{n} \rangle$$
$$+ \frac{1}{2}\int_\Omega |\nabla U|^2 \operatorname{div}\vec{F} - \int_\Omega \mathrm{Jac}_{\vec{F}}[\nabla U, \nabla U] = 0$$

We now make a specific choice of $\vec{F}$ due to Hörmander (50s) $\vec{F}$ is such that $\vec{F}\big|_M = \vec{n}$

Then $\langle \vec{F}\big|_M, \vec{n} \rangle = 1$, $\langle \vec{F}, \nabla U \rangle\big|_M = \partial_n U$

$|\nabla U|^2\big|_M = |\nabla_M u|^2 + (\partial_n U)^2$, $\mathcal{D}_0 u = \partial_n U$

(recall $U\big|_M = u$)

Substituting everything in , and using the formula for the quadratic form of $-\Delta_M$:

$$\int_M (-\Delta_M u)u = \int_M |\nabla_M u|^2$$

we obtain

Thm (Hörmander's identity, ~1954, rediscovered in 2018)

Under the conditions above,

$$(\mathcal{D}_0 u, \mathcal{D}_0 u)_{L^2(M)} - (-\Delta_M u, u)_{L^2(M)}$$

$$= \int_\Omega \left(2\,\mathrm{Jac}_{\vec{F}}[\nabla U, \nabla U] - |\nabla U|^2 \operatorname{div} \vec{F} \right)$$