Part II.

Applications of the method of multipliers

What do we do next with the identity

$$(\mathcal{D}_0 u, \mathcal{D}_0 u)_{L^2(M)} - (-\Delta_M u, u)_{L^2(M)}$$

$$= \int_\Omega \left(2\,\mathrm{Jac}_{\vec{F}}[\nabla U, \nabla U] - |\nabla U|^2 \operatorname{div} \vec{F} \right) \;?$$

For a fixed $\vec{F}$, the RHS is a quadratic form in $|\nabla U|$ with bounded coefficients, and thus its modulus is bounded above by $C\|\nabla U\|^2_\Omega = C(\mathcal{D}_0 u, u)_M$

Thus we have

$$\left| (\mathcal{D}_0 u, \mathcal{D}_0 u)_M - (-\Delta_M u, u)_M \right| \leq C\,(\mathcal{D}_0 u, u)_M$$

C may be expressed via some geometric characteristics of Ω and M [Colbois/Gir./Hass'18; Provenzano/Stubbe'19]

A functional analysis problem: given two non-negative self-adjoint operators $\mathcal{A}$, $\mathcal{B}$ in a Hilbert space $\mathcal{H}$ s.t. $|(\mathcal{A}u, \mathcal{A}u)_{\mathcal{H}} - (\mathcal{B}u, u)_{\mathcal{H}}| \leq C(\mathcal{A}u, u)_{\mathcal{H}}$ for all $u \in \mathrm{Dom}(\mathcal{B}) \cap \mathrm{Dom}(\mathcal{A}^2)$, what can be said about the eigenvalues $\{\alpha_j\}$ of $\mathcal{A}$ and $\{\beta_j\}$ of $\mathcal{B}$?

Thm [Girouard/Karpukhin/ML/Polterovich '22] We have

$$|\alpha_k - \sqrt{\beta_k}| \leq C \quad \forall k \in \mathbb{N}.$$

Idea of proof! Use variational principles for both operators with test spaces constructed of eigenfunctions of the other operator

In our case, this implies that

$$\left| \sigma_k(\Omega) - \sqrt{\lambda_k(-\Delta_M)} \right| \le C \qquad \forall k$$

Steklov e.v. e.v. of the boundary Laplacian

Uniform bound! Recall that for the disk $C=0$

Remarks:

- Smoothness assumptions on $M=\partial\Omega$ may be relaxed to $C^{2,\alpha}$, $\alpha>0$ for $d>2$ and $C^{1,1}$ for planar domains
- Analogues exist for domains on manifolds
- Will be important later for asymptotics of σ_k for large k.

We'll take a slight detour now to discuss the paramer-dependent DtN map $\mathcal{D}_\Lambda$.

Given $\underset{\substack{\cap \\ \mathbb{R}}}{\Lambda} \notin \mathrm{Spec}\left(-\Delta^{\mathrm{Dir}}_{\Omega}\right)$ and $u \in H^{1/2}(\partial\Omega)$,

we can uniquely solve a non-homog. problem

$$\begin{cases} -\Delta U = \Lambda U & \text{in } \Omega \\ U|_M = u & \text{on } M = \partial\Omega \end{cases}$$

$U := \mathcal{E}_\Lambda u$ is called Λ-Helmholtz extension of u

Then $\mathcal{D}_\Lambda : H^{1/2}(M) \longrightarrow H^{-1/2}(M)$ is defined by

$$\mathcal{D}_\Lambda u = (\partial_n U)|_M = (\partial_n \mathcal{E}_\Lambda u)|_M$$

Main facts about $\mathcal{D}_\Lambda$:

- self-adjoint, with discrete real spectrum

$$\sigma_{\Lambda,1} \le \sigma_{\Lambda,2} \le \dots \le \sigma_{\Lambda,k} \le \dots \nearrow +\infty$$

- $$(\mathcal{D}_\Lambda u, u)_{L^2(M)} = \|\nabla U\|^2_{L^2(\Omega)} - \Lambda \|U\|^2_{L^2(\Omega)}$$

- variational principle

$$\sigma_{\Lambda,k} = \min_{\substack{\tilde{\mathcal{L}}\subset H^{1/2}(M)\\ \dim\tilde{\mathcal{L}}=k}} \max_{\substack{u\in\tilde{\mathcal{L}}\\ u\ne0}} \frac{(\mathcal{D}_\Lambda u,u)}{\|u\|^2_{L^2(M)}} = \min_{\substack{\mathcal{L}\subset \mathcal{H}_\Lambda(\Omega)\\ \dim \mathcal{L} = k}} \max_{\substack{U\in\mathcal{L}\\ U\ne0}} \frac{\|\nabla U\|^2_\Omega - \Lambda\|U\|^2_\Omega}{\|U|_M\|^2_M}$$

$\mathcal{H}_\Lambda(\Omega) := \{U\in H^1(\Omega) : -\Delta U - \Lambda U = 0\}$

can be repl. here by $H^1(\Omega)$ if $\Lambda < \lambda_1^D(\Omega)$

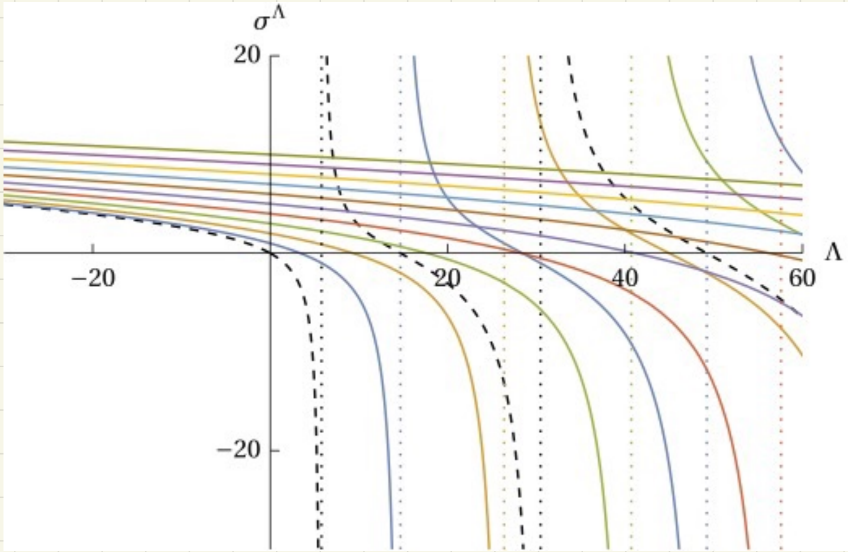
- definition of $\mathcal{D}_\Lambda$ can be extended to $\Lambda \in \operatorname{Spec}(-\Delta_\Omega^{\mathrm{Dir}})$ if we restrict its domain to the orthogonal complement of the subspace of normal derivatives of the corresponding Dirichlet eigenspace
- DtN – Robin duality:

$$\sigma \in \operatorname{Spec}(\mathcal{D}_\Lambda) \Longleftrightarrow \Lambda \in \operatorname{Spec}(-\Delta_\Omega^{\mathrm{Robin}, -\sigma})$$

with the same multiplicities!

[Friedlander '91; Arendt/Mazzeo '12; Hassannezhad/Sher '22
+ history]

Example: Eigenvalues of $\mathcal{D}_\Lambda$ for the unit disk as functions of Λ



Exercise: show that in this case the spectrum of $\mathcal{D}_\Lambda$ consists of eigenvalues

$$\begin{cases} I_m'(-\sqrt{\Lambda})/I_m(-\sqrt{\Lambda}), & \Lambda<0 \\ m, & \Lambda=0 \\ J_m(\sqrt{\Lambda})/J_m(\sqrt{\Lambda}), & \Lambda>0 \end{cases}$$

single for $m=0$
double for $m>0$

Observations

- eigenvalue curves cross $\sigma=0$ when $\Lambda\in \mathrm{Spec}(-\Delta^{Neu})$
- eigenvalue curves "blow up" from $-\infty$ to $+\infty$ when $\Lambda\in \mathrm{Spec}(-\Delta^{Dir})$.
- eigenvalue are monotone decreasing in Λ otherwise

These observations can be turned into rigorous thms not only for a disk but for a general Lipschitz Euclidean domain or a Riem. manifold with boundary, and in particular imply

Thm. [Friedlander; Arendt-Mazzeo]

$$W^{Neu}_{\Omega}(\Lambda) - W^{Dir}_{\Omega}(\Lambda) = W^{\mathcal{D}_\Lambda}(0) \quad \forall \Lambda \in \mathbb{R}$$

number of negative eigenvalues of $\mathcal{D}_\Lambda$

Corollary. If $\Omega \subset \mathbb{R}^d$, then $\lambda^{Neu}_{k+1} < \lambda^{Dir}_{k} \quad \forall k \in \mathbb{N}$

Proof uses the fact that $W^{\mathcal{D}_\Lambda}(0) \geq 1$ for $\Lambda > \lambda^{Dir}_1$

(alternative elementary proof by [Filonov'04])

(the corollary may not hold in Riemannian case)

We now ask the following question:

Can we compare the eigenvalues of $\mathcal{D}_\Lambda(\Omega)$ with those of $-\Delta_{\partial\Omega}$ for (some) $\Lambda \neq 0$?

We have

Thm Let $\Omega \subset \mathbb{R}^d$ be a bdd domain with smooth bdry $M = \partial\Omega$. Then for $\Lambda \leq 0$ we have

$$\left| \sigma_k^\Lambda - \sqrt{\lambda_k(-\Delta_M) - \Lambda} \right| \leq C \qquad k \in \mathbb{N}$$

with some constant C uniformly in both k and Λ

Ideas of proof: [GKLP'22]

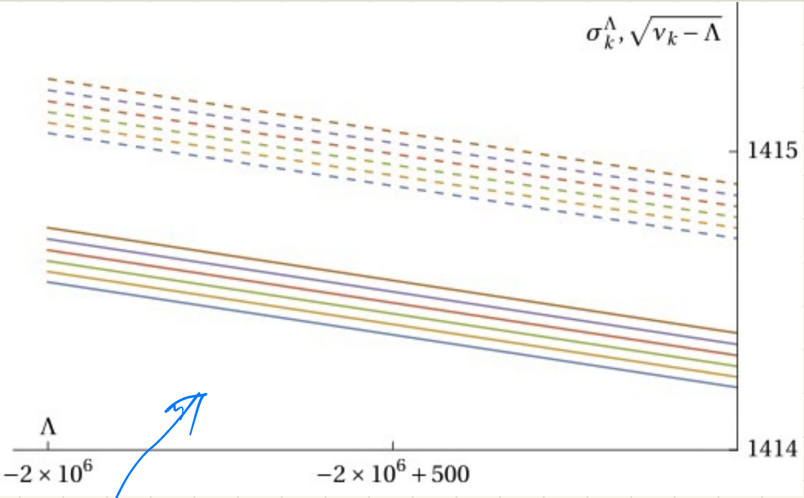
- A variant of generalised Pohozhaev's identity for Hemholtz [Hassannezhad Siffert'20]
- Hence a variant of generalised Hörmander's inequality

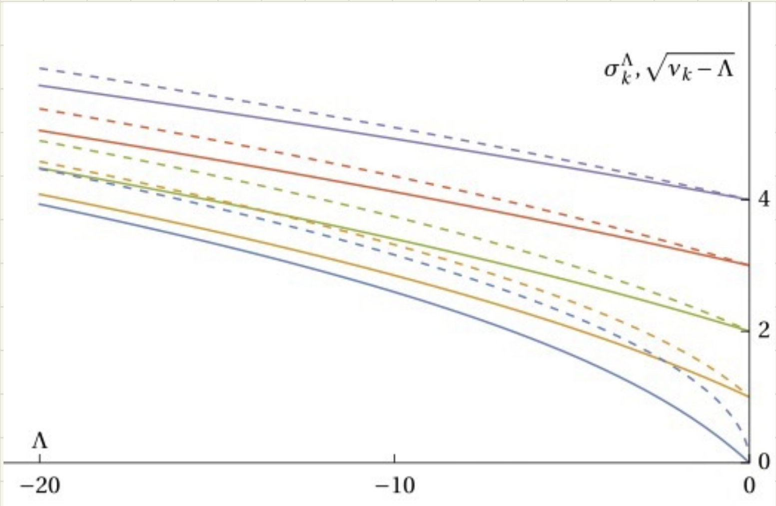
$$\left|(\mathcal{D}_\Lambda u, \mathcal{D}_\lambda u)_M - ((-\Delta_M - \Lambda)u, u)_M\right| \leq C(\mathcal{D}_\Lambda u, u)_M$$

- Use abstract bound with $\mathcal{A} = \mathcal{D}_\Lambda$, $\mathcal{B} = -\Delta_M - \Lambda$.

Remark We will see later that no analogue of this result may hold if boundary has corners

Illustration: the unit disk





solid curves: σ_k^Λ

dashed curves: $\sqrt{\lambda_k^{-\Delta_M} - \Lambda}$

$k \in \{100, 102, 104, 106, 108\}$

$k \in \{1, 3, 5, 7, 9\}$

Before we proceed, some further references (full bibliography will appear at the end of the last set of slides)

[Chandler-Wilde/Graham/Langdon/Spence '12] for a historical overview of the method of multipliers

[Hassell Tao '02] for applications to bounds of $\|\partial_n u_j^{Dir}\|^2_{L^2(\partial\Omega)}$ on normal derivatives of Dirichlet e.f.s

[Rudnick Wigman Yesha '21] for bounds on $\|u_j^{Rob,\gamma}\|^2_{L^2(\partial\Omega)}$ on traces of Robin e.f.s.

etc, etc, ...

Part $\overline{\underline{\text{III}}}$.

Spectral asymptotics for the DtN map $\mathcal{D}_0$

For the rest of this course we will be looking at the asymptotic behaviour of eigenvalues of the Steklov problem on $\Omega \subset \mathbb{R}^d$ [mostly $d=2$] (the DtN map $\mathcal{D}_\Omega$), both in terms of asymptotics of eigenvalues

σ_k, $k \nearrow +\infty$ and the counting function

$$\mathcal{N}^S(\sigma) := \{k : \sigma_k \le \sigma\}$$ as $\sigma \nearrow +\infty$

I'll start by listing some relatively well-known facts.

- Suppose that Ω has a smooth boundary M.

Then $\mathcal{D}_0$ is an elliptic pseudodifferential operator of order 1. Its principal symbol is given by $|\xi|$ and coincides with that of $\sqrt{-\Delta_M}$. Hence these two operators have the same leading term Weyl's asymptotics, and

$$\mathcal{N}^S(\sigma) = C_{d-1}\,|M|_{d-1}\,\sigma^{d-1} + o(\sigma^{d-1})$$

$\nearrow$ Weyl constant

as $\sigma \to +\infty$

- By our previous comparison result, the same asymptotics holds if $\partial\Omega$ is not C^∞ but smooth enough
- On the other hand, for C^∞ boundary in the smooth planar case, error term is much better [Rozenblyum '86] [Edward 93 after Guillemin/Melrose]

$$\sigma_k(\Omega) = \sigma_k(\Omega^*) + O(k^{-N}) \quad \forall N$$

$$k \to +\infty$$

Ω^* — disk with the same perimeter as Ω

- For general Lipschitz domains in $d \geqslant 3$ the one-term Weyl's asymptotics is still an open conjecture. In $d=2$ it was proved very recently by different techniques [Karpukhin Lagacé Polterovich '22]

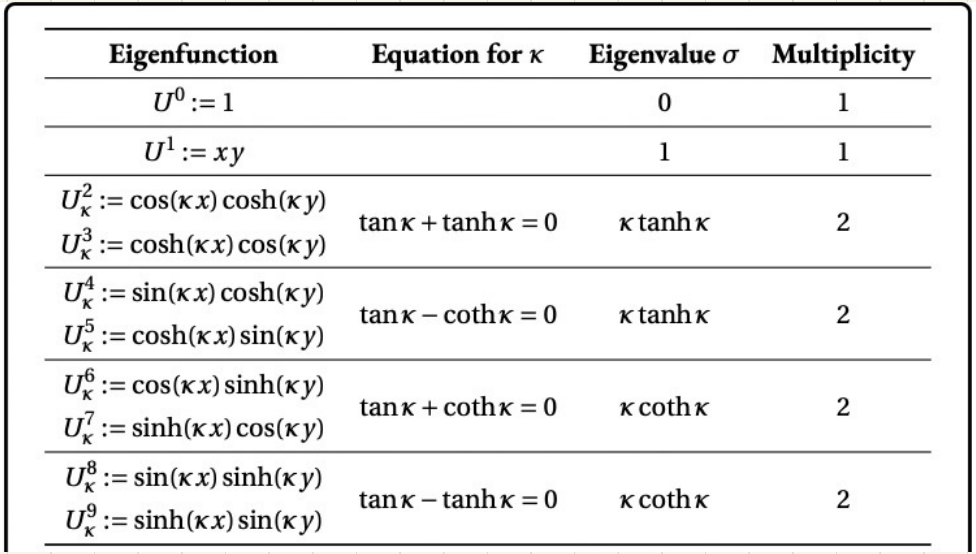
So the question is: what is happenning for "not so smooth" planar domains, say polygons, and can we improve Weyl's asymptotics there?

We start with a seemingly simple

Example. $\Omega = (-1,1)^2$ a square.

[Girouard/Polterovich'17]

We try to find eigenfunctions by separation of variables which gives us



Eigenfunction	Equation for κ	Eigenvalue σ	Multiplicity
$U^0 := 1$		0	1
$U^1 := xy$		1	1
$U^2_\kappa := \cos(\kappa x)\cosh(\kappa y)$ $U^3_\kappa := \cosh(\kappa x)\cos(\kappa y)$	$\tan\kappa + \tanh\kappa = 0$	$\kappa\tanh\kappa$	2
$U^4_\kappa := \sin(\kappa x)\cosh(\kappa y)$ $U^5_\kappa := \cosh(\kappa x)\sin(\kappa y)$	$\tan\kappa - \coth\kappa = 0$	$\kappa\tanh\kappa$	2
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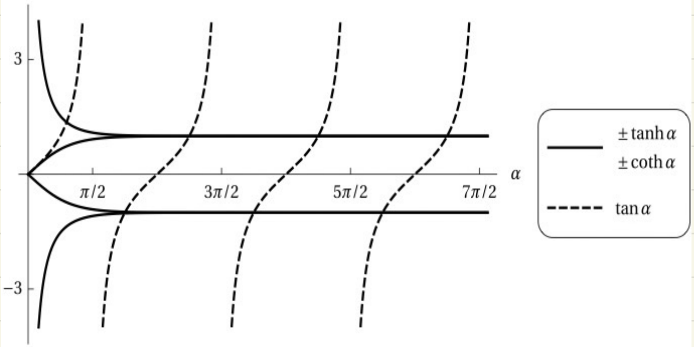


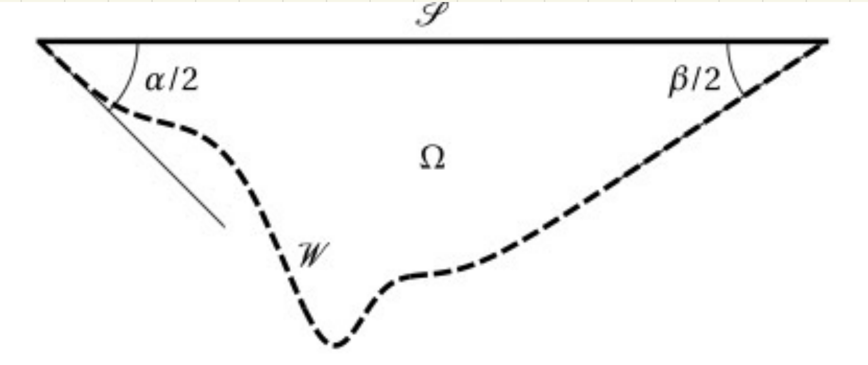
each intersection of a dotted curve with a solid curve gives a double e.v.

But how do we prove that we have found all the eigenvalues and haven't missed any?

To do this, we have to take two sidesteps.

Sloshing problem

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ \partial_n U = \sigma U & \text{on } \mathcal{S} \\ \partial_n U = 0 & \text{on } \mathcal{W} \end{cases}$$



19th century hydrodynamics!

instead of pure Neumann on $\mathcal{W}$ we may decompose

$$\mathcal{W} = \mathcal{W}_N \sqcup \mathcal{W}_D$$

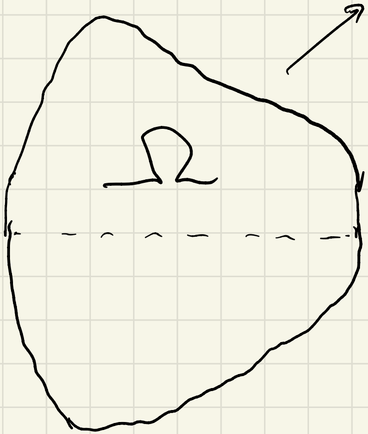
and impose Neumann here Dirichlet

Whatever reasonable b.c. we impose on $\mathcal{W}$, we always have

- the spectrum of the sloshing problem (or another mixed Steklov-Dirichlet-Neumann problem) is discrete and non-negative
- the eigenfunctions restricted to $\mathcal{S}$ form a basis in $L^2(\mathcal{S})$

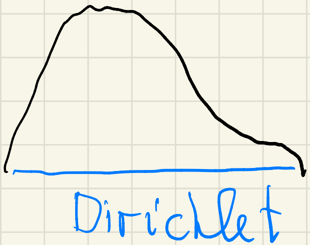
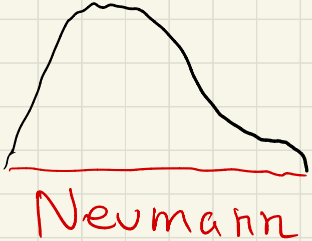
Sloshing problem will re-appear later!

The second trick we need is the symmetry reduction

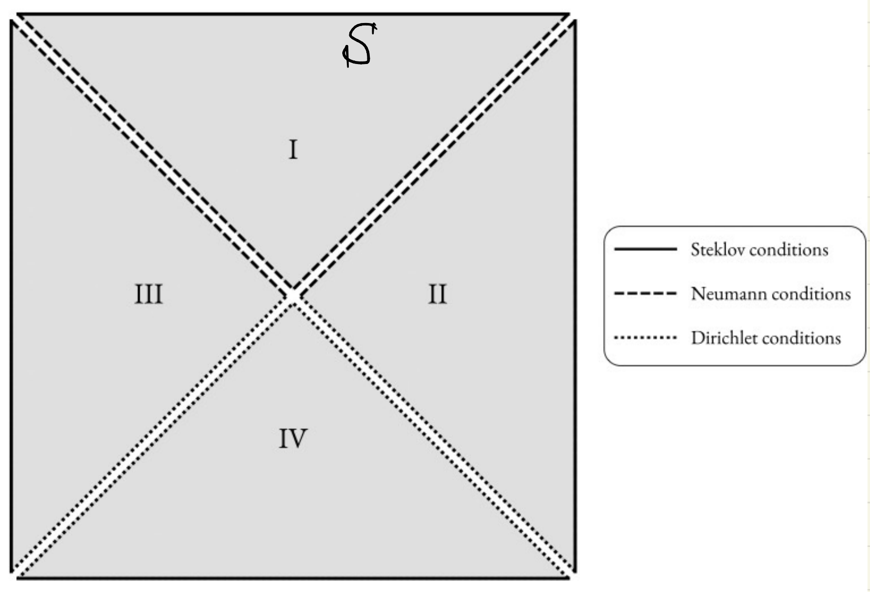


domain with a hyperplane of symmetry, problem and b.c. symmetric

then every eigenfunction is either antisymmetric or symmetric

Spec of = Spec () ∪ Spec ()

Returning now to our Steklov problem on a square: using diagonals as lines of symmetry; we get



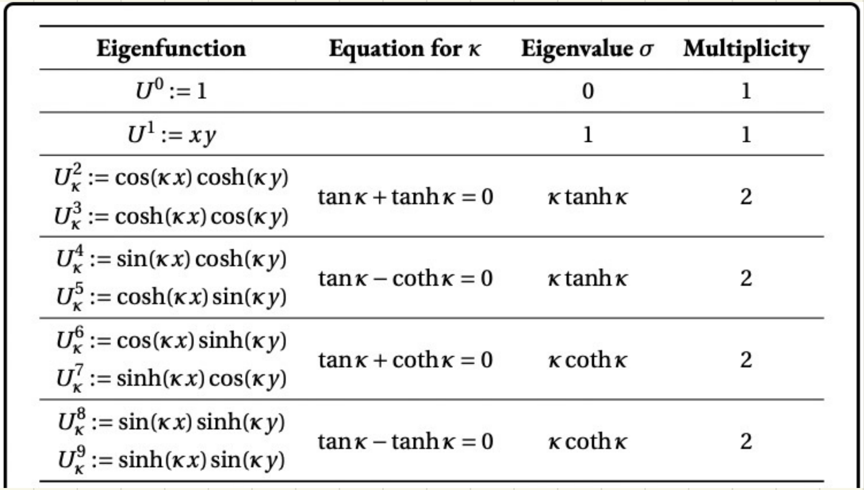
four mixed Steklov-Dir-Neumann problems in 45°-45°-90° triangles.

Look at problem I:

from found eigenfunctions of the square we construct e.fs $U^0, U^1, U^2_\varkappa + U^3_\varkappa, U^8_\varkappa + U^9_\varkappa$ for it.

Why is it the full set? Their traces on S form the full set of e.f. of $f^{(IV)}(x) = \varkappa^4 f(x)$, $f''(\pm 1) = f'''(\pm 1) = 0$ and therefore the basis in $L^2(S)$. Now repeat for $\underline{II}, \underline{III}, \underline{IV}$...

So, we know that all eigenvalues of Steklov on $[-1,1)^2$ are given by



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Corollary Steklov eigenvalues of this square satisfy

$$\sigma_{4m-k} = \left(m-\frac{1}{2}\right)\frac{\pi}{2} + O(m^{-\infty})$$

$m \in \mathbb{N}$, $k \in \{0,\dots,3\}$ as $m \nearrow \infty$

Eigenvalues asymptotically come in clusters of 4

Q: Is it because we have 4 (equal) sides?

Would they appear in clusters of 5 for a regular pentagon?

Part IV.

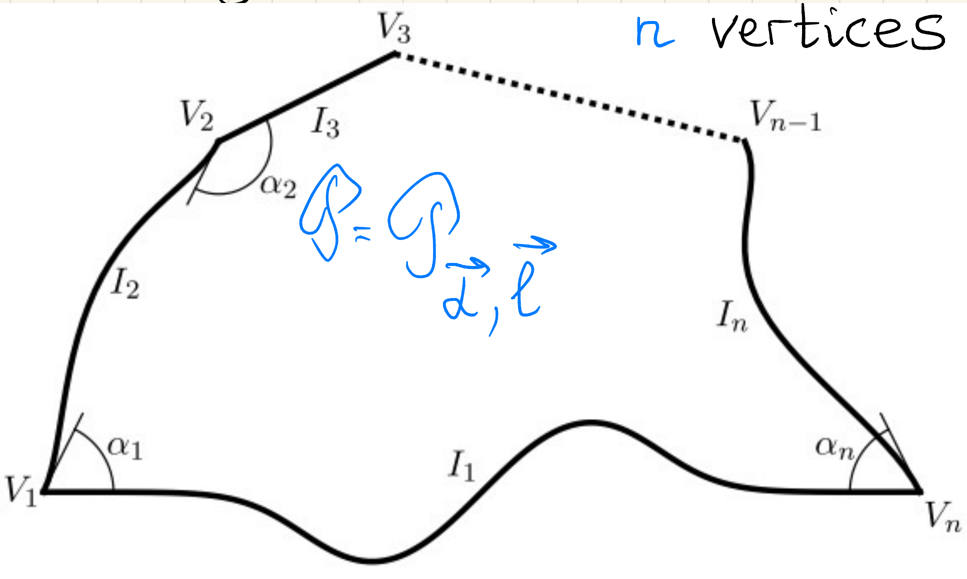
Asymptotics of Steklov eigenvalues in curvilinear polygons

Most of the material in this part is covered by two long papers

ML + Parnovski + Polterovich + Sher J. d'Anal. Math. 2021
Proc LMS 2022

but notation here is slightly different

Setting:



n vertices

$\mathcal{P} = \mathcal{P}_{\vec{\alpha},\vec{\ell}}$

$$\begin{cases} \Delta U = 0 & \text{in } \mathcal{P} \\ \partial_n U = \sigma U & \text{on } \partial\mathcal{P} \end{cases}$$

curvilinear polygon

$\mathcal{P}_{\vec{\alpha},\vec{\ell}}$

vector of angles

$\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$,

$0 < \alpha_j < \pi$

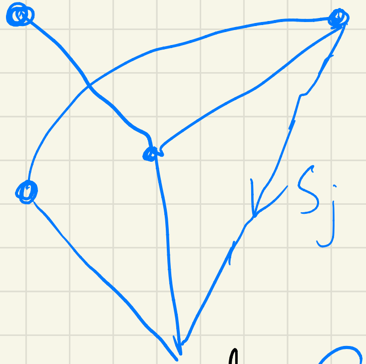
vector of sidelengths

$\vec{\ell} = (\ell_1, \dots, \ell_n)$

side I_j of length ℓ_j joins vertices V_{j-1} and V_j

$L = |\partial\mathcal{P}| = \ell_1 + \dots + \ell_n$

Before stating the main result on the asymptotics of $G_m(\mathcal{G}_{\vec{\alpha},\vec{\ell}})$, $m \nearrow +\infty$ I need another side-step into the theory of quantum graphs [Berkolaiko Kuchment '13]



metric graph G

↑

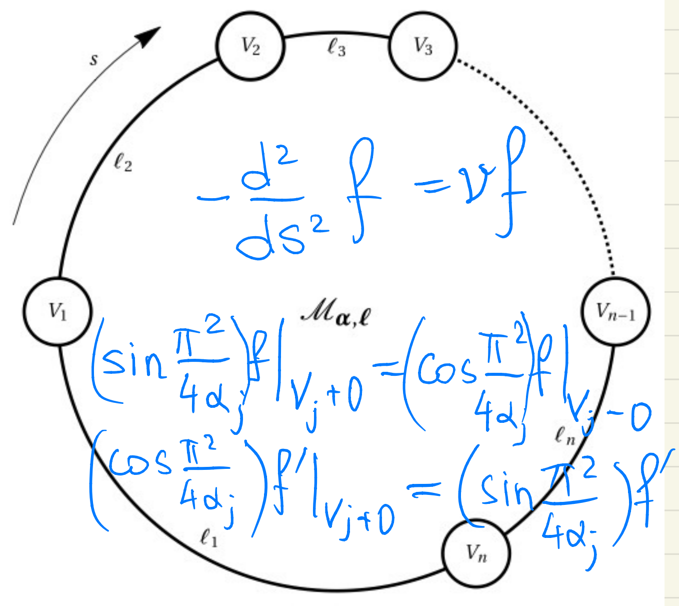
edges E_j have lengths

$$-\Delta_G := \bigoplus \left(-\frac{d^2}{ds_j}\right)\Big|_{E_j}$$

plus some self-adjoint boundary/matching conditions

Looks deceptively simple — in fact, deep subject!

Our main result, philosophy: Once more, compare the Steklov eigenvalues to those of a "boundary operator" but this time the boundary operator is a quantum graph associated with a curvilinear polygon $\mathcal{G}_{\vec{\alpha},\vec{\ell}}$



$$-\frac{d^2}{ds^2} f = \nu f$$

$$\left(\sin\frac{\pi^2}{4\alpha_j}\right) f\Big|_{V_j+0} = \left(\cos\frac{\pi^2}{4\alpha_j}\right) f\Big|_{V_j-0}$$

$$\left(\cos\frac{\pi^2}{4\alpha_j}\right) f'\Big|_{V_j+0} = \left(\sin\frac{\pi^2}{4\alpha_j}\right) f'\Big|_{V_j-0}$$

Thm Let $\mathcal{G}_{\vec{\alpha},\vec{\ell}}$ be a curvilinear polygon, σ_m its Steklov e.v.s and ν_m – e.v.s of QG $\mathcal{M}_{\vec{\alpha},\vec{\ell}}$

Then

$$\sigma_m = \sqrt{\nu_m} + O(m^{-\varepsilon}) \qquad m \nearrow +\infty$$

for some $\varepsilon > 0$

As the QG $\mathcal{M}_{\vec{\alpha},\vec{\ell}}$ depends only on $\vec{\alpha}$ and $\vec{\ell}$, we have

Corollary If Ω^{I}, Ω^{II} are two curvilinear polygons with the same angles and sidelengths taken in the same order then

$$|\sigma_m^{I} - \sigma_m^{II}| = O(m^{-\varepsilon}) \qquad m \nearrow +\infty$$

Defn From now on, the numbers $\tau_m := \sqrt{\nu_m}$ are called the quasi-eigenvalues of the Steklov problem on $\mathcal{G}$.