We haven't done much so far: just replaced our Steklov problem by another (maybe, slightly easier) problem. Q: What can be said about quasi-eigenvalues τ_m?

Thm A number $\tau \geq 0$ is a quasi-eigenvalue of a polygon $\mathcal{P}_{\vec{\alpha}, \vec{\ell}}$ iff it is a root of a particular trigonometric fn $F_{\vec{\alpha}, \vec{\ell}}(\tau)$ next slides

$$\begin{cases} \text{multiplicity of quasi-ev } \tau > 0 = \text{mult. of } \tau \text{ as a root of } F \\ \text{mult. of q.-e. } \tau = 0 \text{ (if present)} = \frac{1}{2}\left(\text{mult. of } \tau \text{ as a root of } F\right) \end{cases}$$

Constructing trig function $F_{\vec{\alpha},\vec{\ell}}(\tau)$

$$\mathcal{Z}^n := \{\pm 1\}^n$$

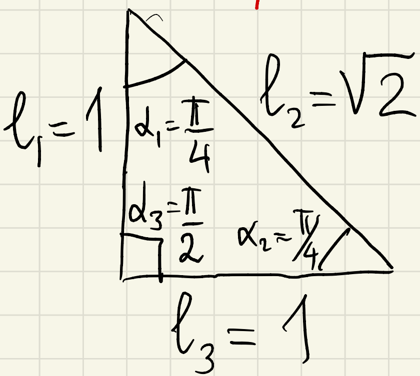
$$F_{\vec{\alpha},\vec{\ell}} = \sum_{\substack{\vec{\xi}\in\mathcal{Z}^n \\ \xi_1 = 1}} \rho_{\vec{\xi}} \cdot \cos(\langle \vec{\ell}, \vec{\xi}\rangle \tau) - \prod_{j=1}^{\infty} \sin\frac{\pi^2}{2\alpha_j}$$

$Ch(\vec{\xi}) = \{ j \in \{1,\dots,n\} \mid \xi_j \neq \xi_{j+1} \}$ set of sign-changes in $\vec{\xi}$

e.g. $Ch((1,1,1)) = \emptyset$, $Ch((-1,-1,1,1)) = \{2,4\}$

$$\rho_{\vec{\xi}} := \prod_{j\in Ch(\vec{\xi})} \cos\frac{\pi^2}{2\alpha_j}$$

Example



$\vec{\xi}$	$\langle \vec{\ell}, \vec{\xi} \rangle$	$Ch(\xi)$	$\rho_{\vec{\xi}}$
$(+,+,+)$	$2+\sqrt{2}$	$\emptyset$	1
$(+,+,-)$	$\sqrt{2}$	$\{2,3\}$	-1
$(+,-,+)$	$2-\sqrt{2}$	$\{1,2\}$	1
$(+,-,-)$	$-\sqrt{2}$	$\{1,3\}$	-1

Also $\prod_{j=1}^{3} \sin \frac{\pi^2}{2\alpha_j} = 0$

Thus $F_{\vec{\alpha},\vec{\ell}}(\tau) = \cos((2+\sqrt{2})\tau) - 2\cos(\sqrt{2}\tau) + \cos((2-\sqrt{2})\tau)$

$= -4(\cos^2\tau - 1)\cos(\sqrt{2}\tau) \Rightarrow$ we have

- a single qe $\tau = 0$
- double qe's at $\tau = \pi m,\ m \in \mathbb{N}$

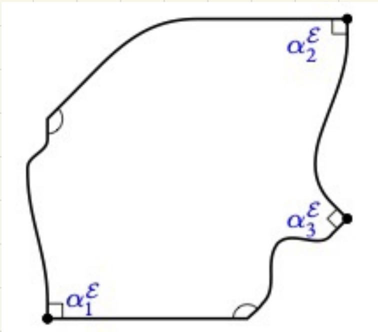
- single qe's at $\tau = \frac{\pi}{\sqrt{2}}\left(m - \frac{1}{2}\right)$, $m \in \mathbb{N}$

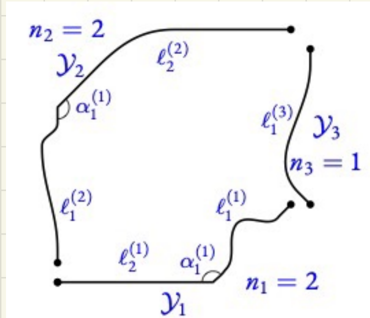
More can be said about the asymptotics of Steklov eigenvalues and eigenfunctions

- As we will see, a particular role is played by some arithmetic properties of the angles, specifically by the presence or absense of the exceptional angles from the set $\mathcal{E} := \left\{ \frac{\pi}{2k}, k \in \mathbb{N} \right\}$
- We'll also mention special angles $\mathcal{S} := \left\{ \frac{\pi}{2k-1}, k \in \mathbb{N} \right\}$
- In either case $\alpha \in \mathcal{E} \cup \mathcal{K}$, the oddity of the angle is defined as $\mathcal{O}(\alpha) = (-1)^k$.

In the presence of exceptional angles $\alpha_1^{\varepsilon}, \dots, \alpha_K^{\varepsilon}$:

we need some re-labelling





- they split the boundary into K exceptional boundary components $\mathcal{Y}_\varkappa$, each with $n_\varkappa$ pieces, $\varkappa = 1, \dots, K$, so $n_1 + \dots + n_K = n$

- $\varkappa^{th}$ component has lengths $\vec{\ell}^{(\varkappa)} = (\ell_1^{(\varkappa)}, \dots, \ell_{n_\varkappa}^{(\varkappa)})$ and angles $\vec{\alpha}^{(\varkappa)} = (\alpha_1^{(\varkappa)}, \dots, \alpha_{n_\varkappa - 1}^{(\varkappa)})$

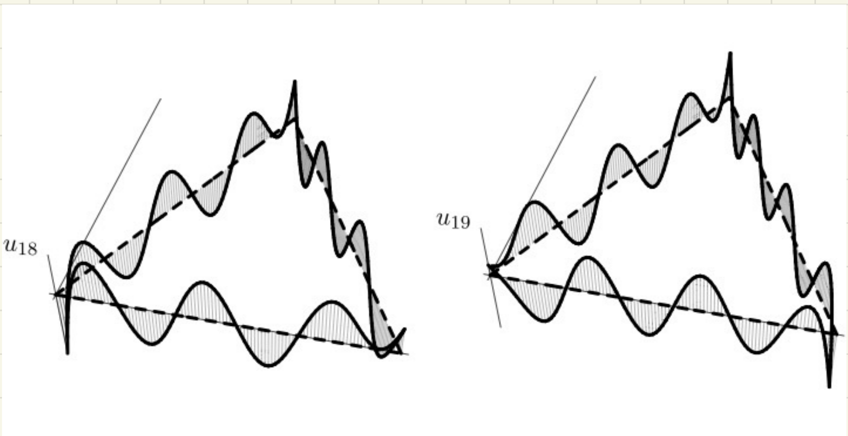
- An exc. boundary component may be $\begin{cases} \underline{\text{even}}, \ \mathcal{O}(\alpha_\varkappa^{\varepsilon}) = \mathcal{O}(\alpha_{\varkappa-1}^{\varepsilon}) \\ \underline{\text{odd}}, \ \mathcal{O}(\alpha_\varkappa^{\varepsilon}) \neq \mathcal{O}(\alpha_{\varkappa-1}^{\varepsilon}) \end{cases}$

Non-exceptional case vs Exceptional case

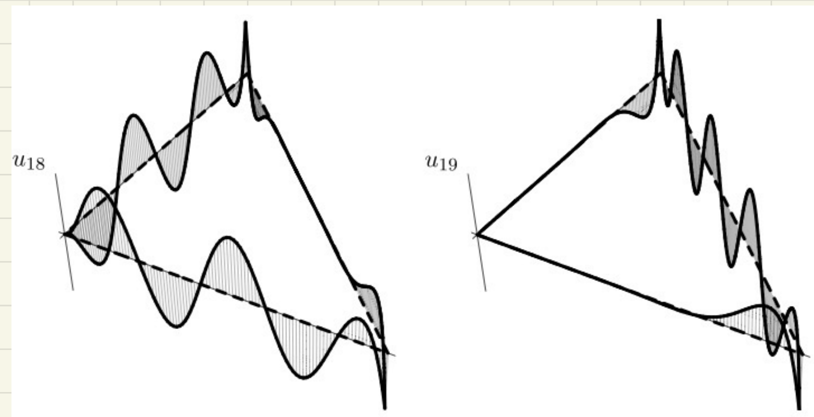
$\sigma=0$	present only if $\prod_{j=1}^{n} \tan\frac{\pi^2}{4\alpha} = 1$; always simple	has multiplicity $\frac{1}{2}K_{odd}$ # of odd except. bdry comps
$\sigma>0$	multiplicity one or two	any multiplicity up to K
$F(\tau)$		factorises: $F_{\vec{\alpha},\vec{e}}(\tau) = \prod_{k=1}^{K} F_{even/odd,\, y_k}(\tau)$
eigen-functions of $\mathcal{D}_0$	"equidistributed" over the whole boundary	concentrate on some exceptional bound. components

In the last line equidistributed means that for any arc $I \subset \partial \mathcal{P}$ (does not have to be a side)

$$\lim_{m\to\infty} \frac{\|U_m\|_{L^2(I)}}{\|U_m\|_{L^2(\partial\mathcal{P})}} = \frac{|I|}{|\partial\mathcal{P}|}$$



non-exceptional case, equilateral Δ-gle

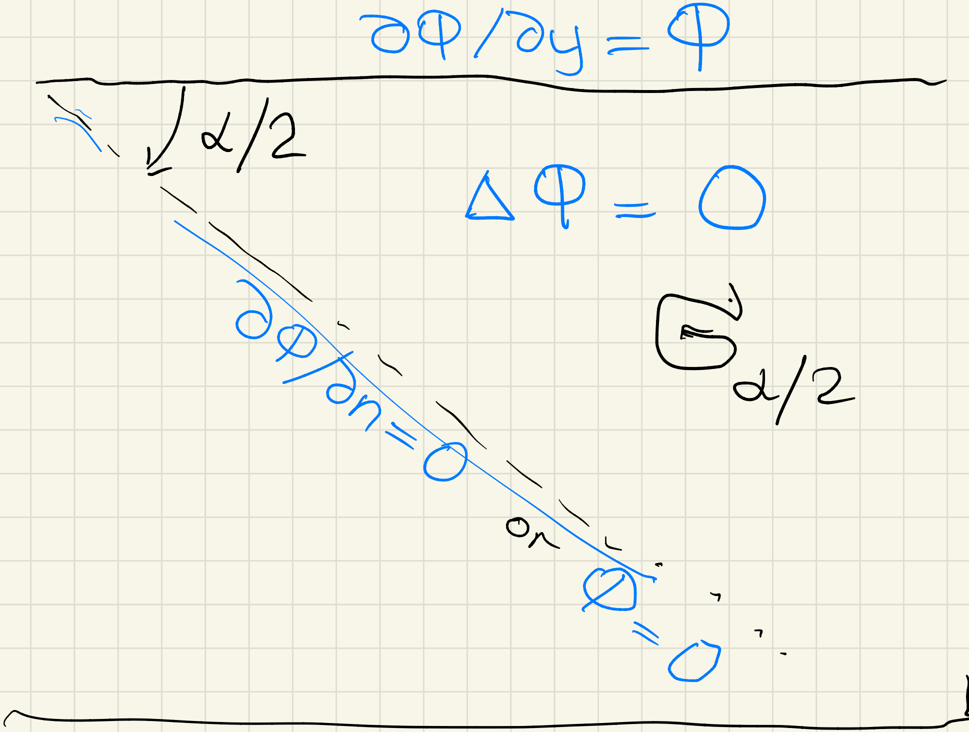


exceptional case, $90^\circ - 45^\circ - 45^\circ$ Δ-gle

MAIN STEPS OF THE PROOFS

- The result about $F_{\vec{\alpha},\vec{\ell}}(\tau)$ is obtained by just writing out the secular equation of our quantum graph $\mathcal{M}_{\vec{\alpha},\vec{\ell}}$: requires some trickery, but generally straight-forward
- The difficult part is about quasi-eigenvalues, and we first need another classical hydrodynamics problem...

The sloping beach problems in an infinite sector



Question:

are there solutions bounded in $\mathfrak{S}_{\alpha/2}$

and such that

$$\Phi(x,y) = \cos(x-\xi)e^{y} + R(x,y)$$

with

$$\rho = \sqrt{x^2+y^2}$$

$$R(x,y) + |\rho\nabla R(x,y)| = O(\rho^{-r}),$$

with some $r > 0$, $\rho \to \infty$

Answers: yes if one takes particular values of constant

$$\xi = \begin{cases} \xi_{\alpha/2,\,Neu} = \frac{\pi}{4} - \frac{\pi^2}{4\alpha} & \text{for Neu on the bottom} \\ \xi_{\alpha/2,\,Dir} = \frac{\pi}{4} + \frac{\pi^2}{4\alpha} & \text{for Dir on the bottom} \end{cases}$$

[originally due to Lewy and Peters 40s-50s, improved/extended by us; we can take $r = \pi/\alpha$ in the remainder]

We will denote the corresponding solutions

$$\Phi_{\alpha/2,\,Neu}(x,y) \quad \text{and} \quad \Phi_{\alpha/2,\,Dir}(x,y)$$

Robin problem in the "full" sector $\mathfrak{S}_\alpha$

$$\begin{cases} \Delta \tilde{\Phi} = 0 \text{ in } \mathfrak{S}_\alpha \\ \partial_n \tilde{\Phi} = \tau \tilde{\Phi} \\ \text{on } I_{in} \cup I_{out} \end{cases}$$

τ large

$\Phi_{symm}(x, y)$ – symmetric extension of $\Phi_{\alpha/2, Neu}(\tau x, \tau y)$

$\Phi_{antis.}(x, y)$ – antisymm. extension of $\Phi_{\alpha/2, Dir}(\tau x, \tau y)$

Consider now a non-trivial linear comb. of Φ_{symm}, $\Phi_{antisym}$

It solves the Robin problem in the full sector.

Moreover, its traces on the boundary rays $I_{in/out}$ behave as

$$\begin{aligned}\tilde{\Phi}\big|_{I_{in}}(s) &= h_{in,1}\, e^{i\tau s} + h_{in,2}\, e^{-i\tau s} + o(1)\\ \tilde{\Phi}\big|_{I_{out}}(s) &= h_{out,1}\, e^{i\tau s} + h_{out,2}\, e^{-i\tau s} + o(1)\end{aligned} \qquad (*)$$

with some vectors $\vec{h}_{in/out} = \begin{pmatrix} h_{in/out,1} \\ h_{in/out,2} \end{pmatrix} \in \mathbb{C}^2$

We call such a solution $\tilde{\Phi}_\tau(x, y; \vec{h}_{in}, \vec{h}_{out})$

a Peters solution

Next question: What should be the relations (if any) between the vectors $\vec{h}_{in}$, $\vec{h}_{out}$ for a Peters soln with given asymptotics to exist?

Some linear algebra + trigonometry gives

Thm • Let $\alpha \notin \mathcal{E}$. Then for every $\vec{h}_{in} \in \mathbb{C}^2$ there exists a Peters soln $\tilde{\Phi}_\tau(x, y; \vec{h}_{in}, \vec{h}_{out})$ if we take

$$\vec{h}_{out} = \underbrace{\begin{pmatrix} \operatorname{cosec} \pi^2/4\alpha & -i\cot \pi^2/4\alpha \\ i\cot \pi^2/4\alpha & \operatorname{cosec} \pi^2/4\alpha \end{pmatrix}}_{= A(\alpha)} \vec{h}_{in}$$

contd...

- let $\alpha = \frac{\pi}{2k} \in \mathcal{E}$, $k \in \mathbb{N}$. Then a Peters soln exists if the vectors $\vec{h}_{in}, \vec{h}_{out}$ satisfy

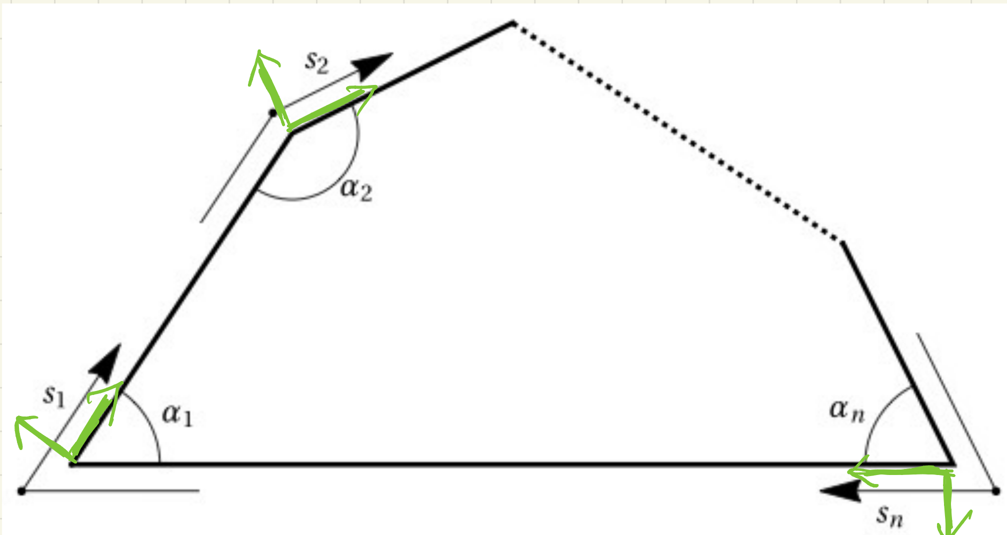
$$\langle \vec{h}_{in}, \vec{X} \rangle_{\mathbb{C}^2} = \langle \vec{h}_{out}, \overline{\vec{X}} \rangle_{\mathbb{C}^2} = 0, \text{ where}$$

$$\vec{X} = \begin{pmatrix} e^{(-1)^{k+1} i\pi/4} \\ e^{(-1)^k i\pi/4} \end{pmatrix}$$

Remarks: - in both cases we choose two of the four quantities $\vec{h}_{in}, \vec{h}_{out} \in \mathbb{C}^2$ but do this differently

- $A(\alpha)$ invertible $\Longleftrightarrow$ $\alpha \notin \mathcal{E}$.

Constructing the quasimodes for a straight polygon: an outline



local boundary coordinate s_n centred at V_n

$\mathcal{V}_j$: sector $V_{j-1}V_jV_{j+1} \longmapsto$ sector S_{α_j}

$(x_j', y_j') = \mathcal{V}_j(x,y)$ local Cart. coordinates

We seek quasimodes of the Steklov problem

$\tilde{U}_\tau(x,y)$ such that near each vertex

$$\tilde{U}_\tau(x,y) = \tilde{\Phi}_\tau\left(x_j', y_j'; \vec{h}_{j,in}, \vec{h}_{j,out}\right)$$

with still unknown

a Peters solution

vectors $\vec{h}_{j,in}, \vec{h}_{j,out}$

For simplicity, assume there are <u>no</u> exceptional angles

Then we must have $\boxed{\vec{h}_{j,out} = A(\alpha_j)\vec{h}_{j,in}}$

We also have $\tilde{U}_\tau|_{\partial\mathcal{G}} = \hat{u} + o(1)$ $\quad \tau \to \infty$

Consider $\tilde{u}\big|_{I_j}$ $\longrightarrow$ side joining V_{j-1} and V_j. It may be written in two ways:

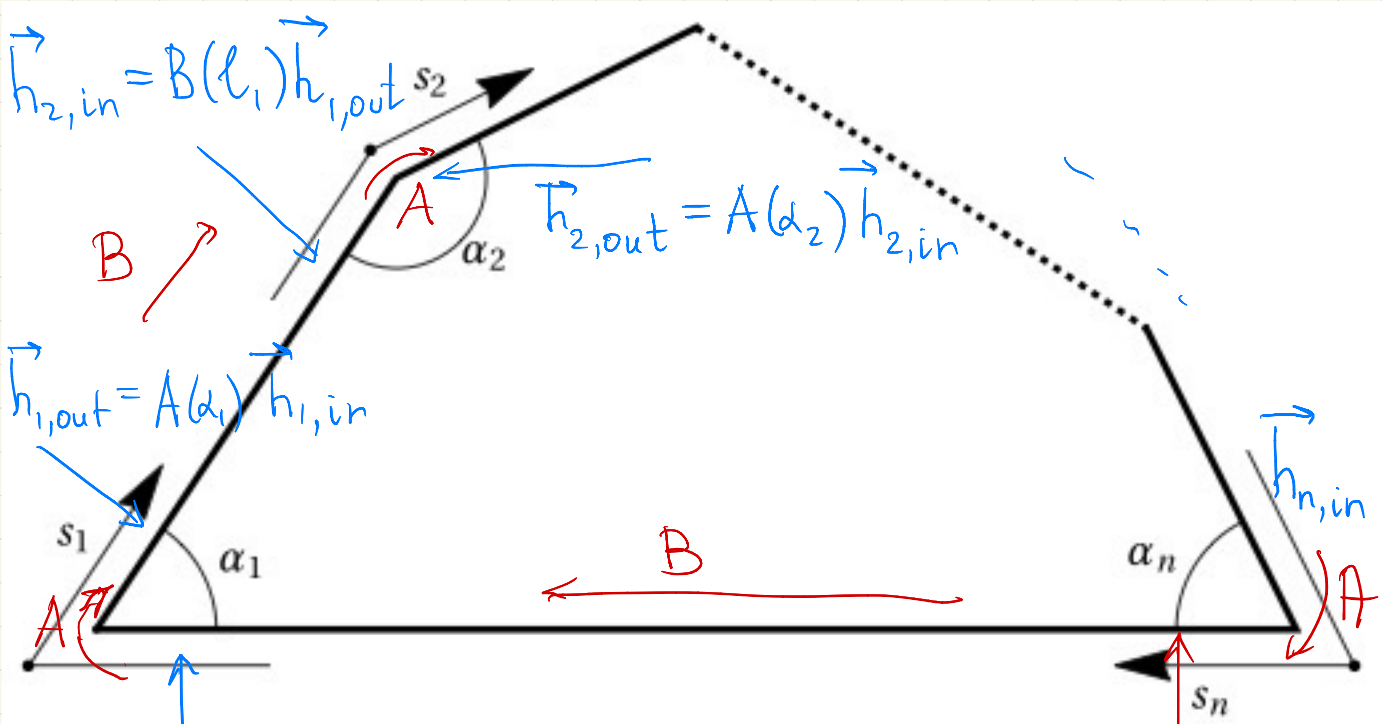
a trigonometric fn in S_j involving vectors $\vec{h}_{j,in}$, $\vec{h}_{j,out}$ (using Peters soln near V_j) $=$ a trig. fn in S_{j-1} using vectors $\vec{h}_{j-1,in}$, $\vec{h}_{j-1,out}$ (using Peters soln near V_{j-1})

A calculation gives

$$\vec{h}_{j,in} = B(\ell_j, \tau)\,\vec{h}_{j-1,out}$$

where $B(\ell,\tau) := \begin{pmatrix} e^{i\ell\tau} & 0 \\ 0 & e^{-i\ell\tau} \end{pmatrix}$

So, what do we get?



$$\vec{h}_{2,in} = B(\ell_1)\vec{h}_{1,out}$$

$$\vec{h}_{2,out} = A(\alpha_2)\vec{h}_{2,in}$$

$$\vec{h}_{1,out} = A(\alpha_1)\vec{h}_{1,in}$$

$$\vec{h}_{1,in} = B(\alpha_1)\vec{h}_{n,out}$$

$\boxed{\text{Start here } \vec{h}_{n,out}}$

Thus, we've arrived at the equation

$$\vec{h}_{n,out} = \underbrace{A(\alpha_n)B(\ell_n,\tau)A(\alpha_{n-1})B(\ell_{n-1},\tau)\cdots A(\alpha_1)B(\ell_1,\tau)}_{:= T(\vec{\alpha},\vec{\ell})}\vec{h}_{n,out}$$

product of $2n$ 2×2 matrices

Therefore the matrix $T(\vec{\alpha},\vec{\ell})$ has an eigenvalue 1

$\Updownarrow$ (easy check)

$$\mathrm{Tr}\,(T(\vec{\alpha},\vec{\ell})) = 2 \iff F_{\vec{\alpha},\vec{\ell}}(\tau) = 0$$

(difficult check)

What remains to be done?

- Rigorous construction of the quasimodes $\tilde{U}_{\tau_m}$ using appropriate cut-offs
- Then it is relatively easy to see that

$$\| \partial_n \tilde{U}_{\tau_m} - \tau_m \tilde{U}_{\tau_m} \|_{L^2(\partial\Omega)} \longrightarrow 0$$

quantatively

so they are indeed quasimodes:

there exists a subsequence of exact Steklov e.v's σ_{i_m} s.t.

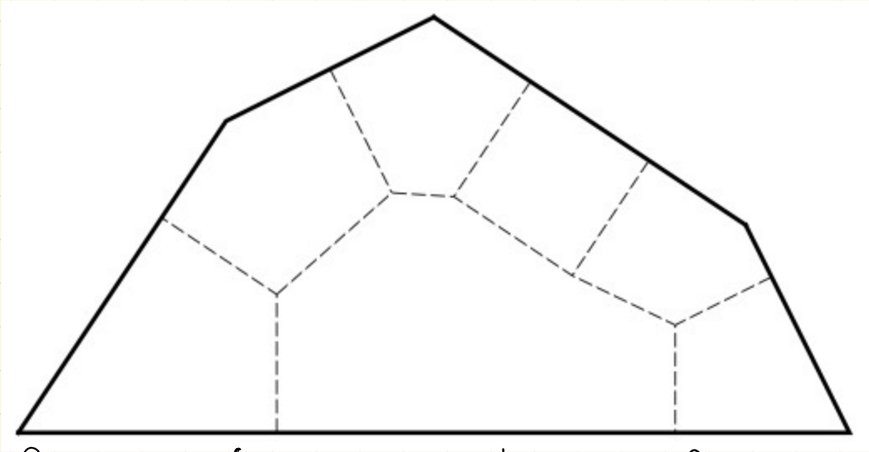
$$|\tau_m - \sigma_{i_m}| = o(1)$$

- the most difficult part: show that

$$i_m = m$$

(in other words, we haven't missed any e.v's)

Done with the help of Dirichlet-Neumann bracketing:

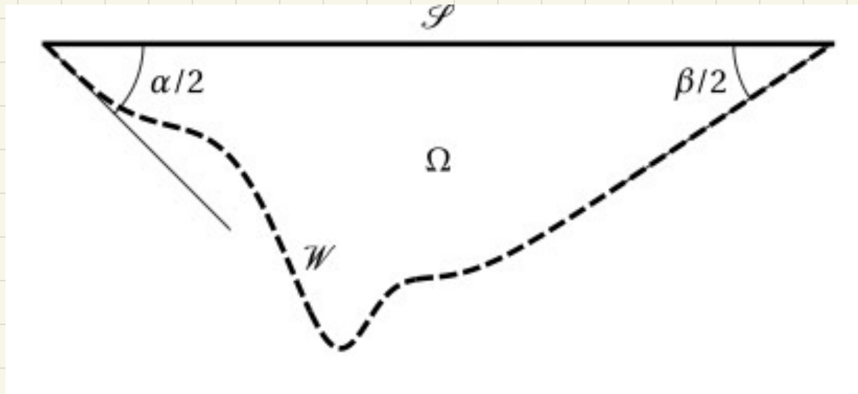


* add cuts with Dirichlet or Neumann (one by one)
* compare with asymptotics of sloshing/other mixed Steklov-Dirichlet-Neumann problems + transplantation tricks

more later

- account for curved boundaries away from the vertices (easy, there is a good ansatz)
- account for curved boundary near the vertices (hard, potential theory in the spirit of Costabel)
- adjust for presence of exceptional angles

Asymptotics of eigenvalues of the sloshing and related problems



$$\mathcal{W} = \mathcal{W}_D \sqcup \mathcal{W}_N$$

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U\big|_{\mathcal{W}_D} = 0 & \partial_n U\big|_{\mathcal{W}_N} = 0 \\ \partial_n U = \sigma U & \text{on } \mathcal{S} \end{cases}$$

Done in the same manner as pure Steklov, by matching solns of the sloping beach problems near the corners

Thm [LPPS'21] For $0 < \frac{\alpha}{2}, \frac{\beta}{2} < \frac{\pi}{2}$

$$|S|\,\sigma_m = \pi\left(m - \frac{1}{2}\right) + \frac{\pi^2}{8}\left(\pm\frac{2}{\alpha} \pm \frac{2}{\beta}\right) + o(1), \quad m \to \infty$$

where $+$ is taken for the Dirichlet condition near the corner and $-$ for Neumann

Remarks • Also works for $\frac{\alpha}{2}, \frac{\beta}{2} = \frac{\pi}{2}$ with some extra geom conds

• If walls are straight near the corners, o-term is better • Proves Conj of Fox-Kuttler '1980s

Some open problems:

- Numerics suggest that our asymptotics "work" for angles $\in [\pi, 2\pi)$ for the Steklov, and $\frac{\alpha}{2}, \frac{\beta}{2} \in [\pi/2, \pi)$ for sloshing etc. We couldn't prove it as our remainder estimate for the sloping beach problem is not good enough.
- Numerics / hand-waving arguments also suggest that the next-order correction in the asymptotics for σ_m should come from the boundary curvature at the vertices

• What happens in 3D and higher? Only basics/specific examples are known so far:

* Steklov: [Ivrii 2019] + for cuboids [Girouard + Lagacé + Polterovich + Savo '19]

* sloshing in a particular prism [Mayrand + Senécal + St-Amant '21]

Remark. Recall that in the smooth case we had the uniform bound $\left|\sigma_{\Lambda,k} - \sqrt{\lambda_k(-\Delta_M) - \Lambda}\right| \leq C$ for eigenvalues of $\mathcal{D}_\Lambda$. Is it possible to have a similar uniform bound for polygons, maybe with $-\Delta_M$ replaced by some other "boundary" operator (independent of Λ). The answer is NO, and it follows from Robin - DtN duality and results on asymptotics of $\sigma_{\Lambda,k}$ as $\Lambda \to -\infty$ for a fixed k derived from that and [Khalile'18], [Khalile/Pankrashkin'18], etc....

Part $\overline{\underline{V}}$.

Inverse Steklov problems

We start with a

Defn Two Riemannian manifolds with bdry, or two Euclidean domains are called **Steklov isospectral** if their Steklov spectra coincide

Two examples from [Girouard / Parnovski / Polterovich / Sher 14]

If M_1, M_2 are isospectral $(-\Delta)$ Riem. manifolds, then $M_j \times (0,h)$ are Steklov isospectral

If g_1 and $g_2 = \rho g_1$ are two conf. equivalent metrics on Ω with $\rho|_{\partial\Omega} = 1$, then (Ω, g_j) are Steklov isospectral

At the same time,

Open Question: do there exist non-isometric Steklov isospectral planar Euclidean domains?

Defn. We say that two domains Ω_I and Ω_{II} are asymptotically (Steklov) isospectral or quasi-isospectral if

$$|\sigma_k(\Omega_I) - \sigma_k(\Omega_{II})| = o(1) \quad k \nearrow \infty$$

Fact Any two simply connected smooth planar domains with the same perimeter are asymptotically Steklov isospectral

Q: Which Steklov spectral invariants do we have, i.e. what geometric information can we recover from Steklov spectrum?

A. In the smooth case, the perimeter + in a non-simply connected case, the number of boundary components and their length [GPPS'14]

We will now concentrate on curvilinear polygons only, a separate set of slides