# Inverse Steklov problem

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with Stanislav Krymski, Leonid Parnovski, Iosif Polterovich, David Sher [IMRN'21]

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GEMSTONE Lectures
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# Direct problem summary

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curvilinear polygon $\mathcal{P}$

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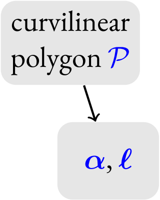

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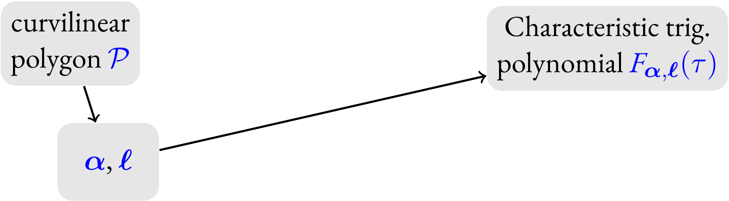

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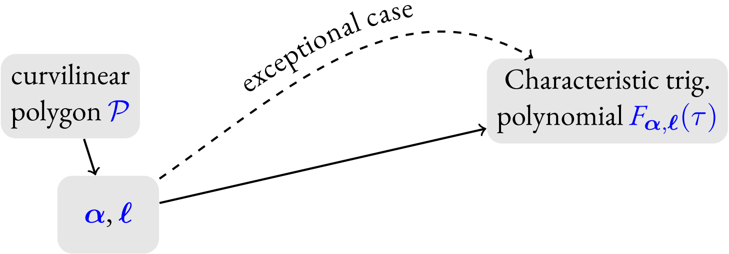

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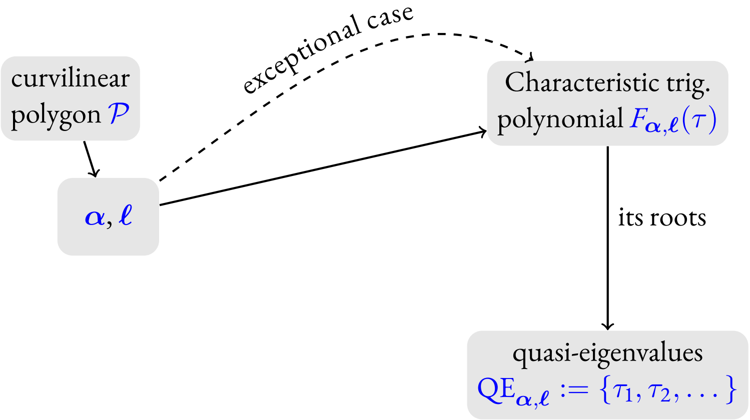

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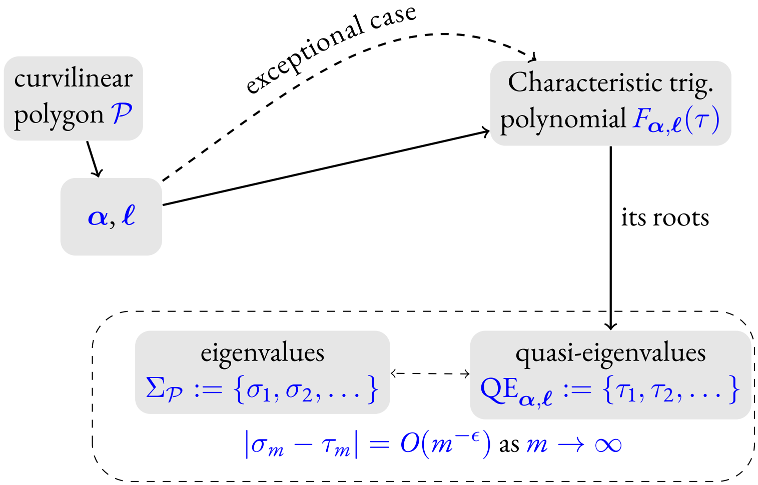

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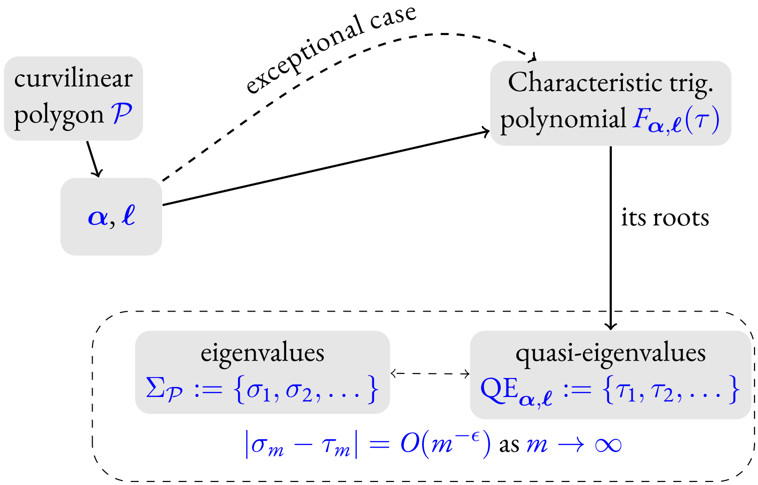

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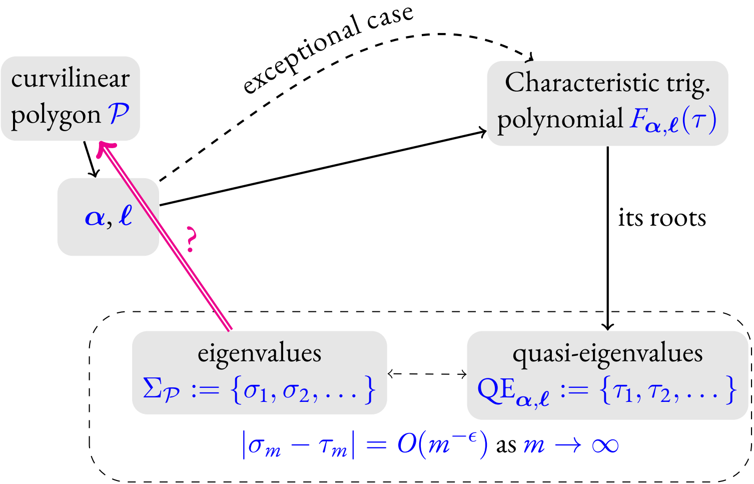

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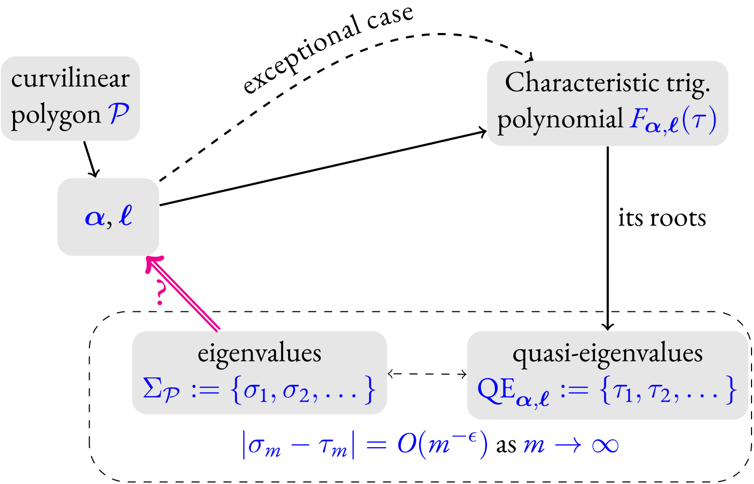

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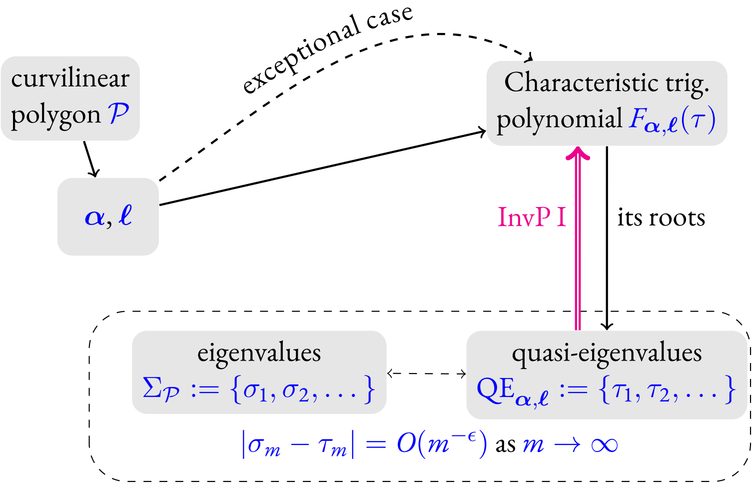

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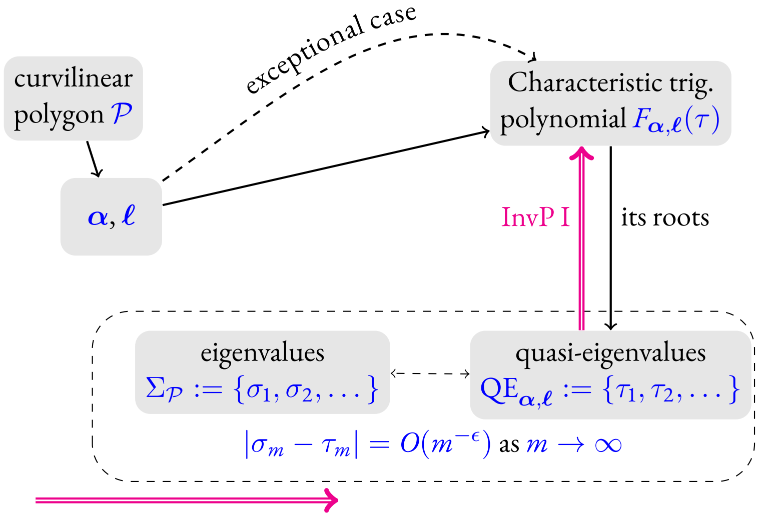


unconditional, pseudo-algorithmic

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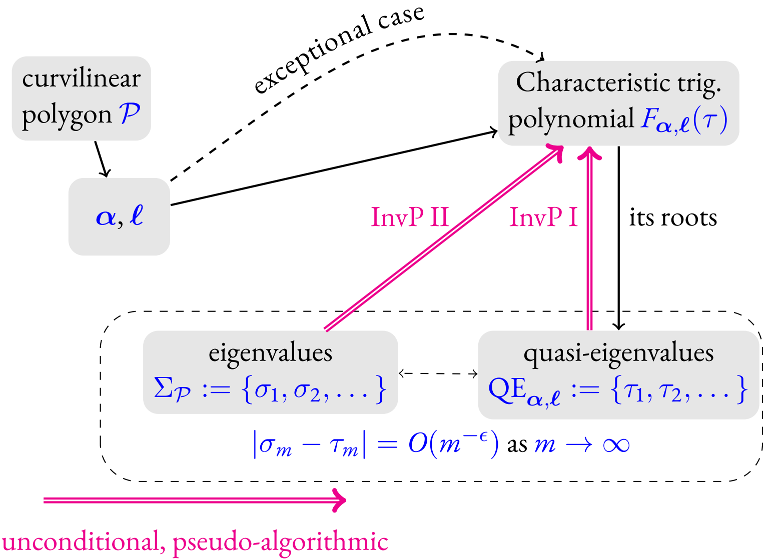

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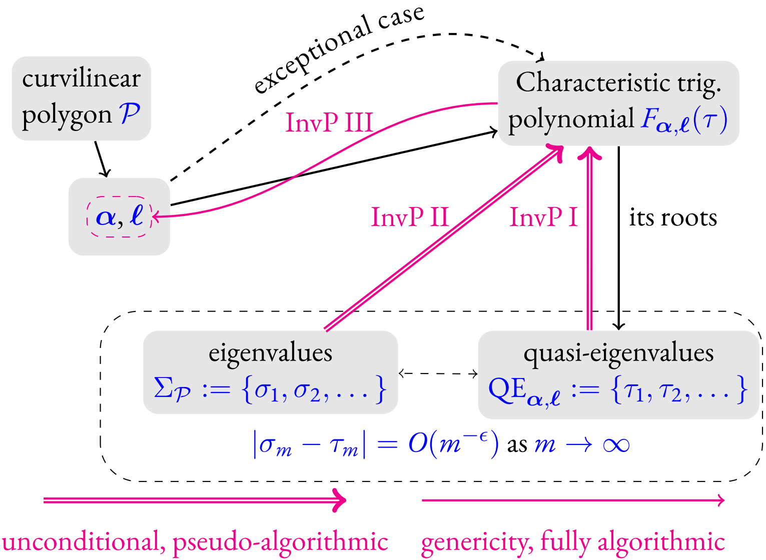

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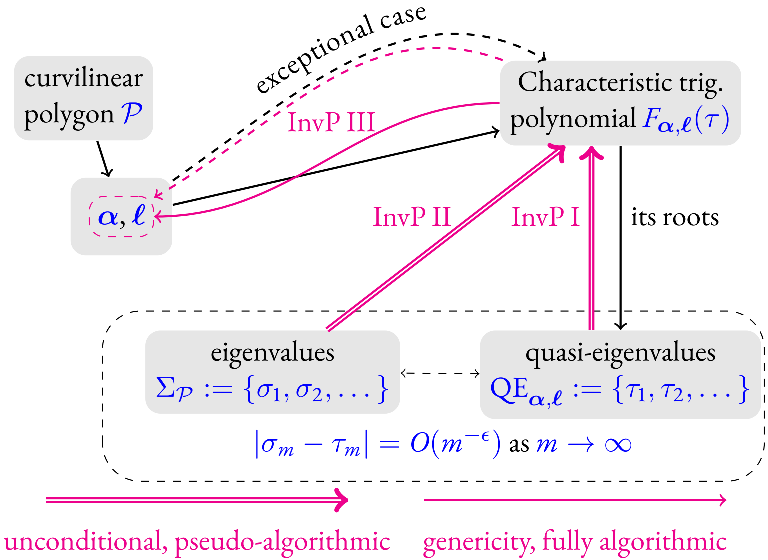

**Definition (Steklov isospectrality and and asymptotically isospectral ity)**

We say that two domains $\Omega_1$ and $\Omega_2$ are (Steklov) isospectral if their Steklov spectra coincide, $\Sigma_{\Omega_1} = \Sigma_{\Omega_2}$.

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(a) any two isospectral planar domains are also asymptotically isospectral ; (b) it is not known if there exist any planar isospectral non-isometric domains; (c) known Steklov spectral invariants are the perimeter (by Weyl's law), and in the smooth case also the number and lengths of connected boundary components [GPPS14]; (d) on the other hand, any two smooth planar simply connected domains with the same perimeter are Steklov asymptotically isospectral , and moreover $o(m^{-\infty})$-asymptotically isospectral .

# Inverse problems — notation and definitions

- For $\boldsymbol{\alpha} \in (0, \pi)^n$, its *cosine vector* is

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## Definition (Loose equivalence)

We say that two curvilinear polygons $\mathcal{P}(\boldsymbol{\alpha}, \boldsymbol{\ell})$ and $\tilde{\mathcal{P}}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\ell}})$ are loosely equivalent if one can choose the orientation and the enumeration of vertices of these polygons in such a way that $\boldsymbol{\ell} = \tilde{\boldsymbol{\ell}}$ and either $\mathbf{c}_{\boldsymbol{\alpha}} = \mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$ or $\mathbf{c}_{\boldsymbol{\alpha}} = -\mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$.

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Subject to admissibility conditions, we have …

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*This can be dropped; the statement becomes slightly more complicated; it will come later*

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*$\boldsymbol{\ell}$ and $\tilde{\boldsymbol{\ell}}$ are the same modulo a cyclic shift and a change of orientation; cosine vectors $\mathbf{c}_{\boldsymbol{\alpha}}$ and $\mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$ are the same up to a change of sign*

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> *This means we have formulae/algorithms for that but some steps may be non-trivial to realise numerically*

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This is just a variant of Hadamard–Weierstrass Factorisation Theorem:

$\inf\left\{r \in \mathbb{R} : f(z) = O\left(e^{|z|^r}\right) \textit{ as } |z| \to \infty\right\} = 1$

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$$f(z)=CQ_\Gamma(z),\quad Q_\Gamma(z):=z^{2m_0}\prod_{\gamma_j\in\Gamma\setminus\{0\}}\left(1-\frac{z^2}{\gamma_j^2}\right).$$

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Hadamard-Weierstrass Theorem immediately gives the result with an extra factor $e^{g(z)}$, where $g(z)$ is linear. But since $f$ is even, so is $g$, which is therefore a constant. □

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Inverse Problem I: from an infinite product to the explicit form of a trigonometric polynomial

Write

$$F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\tau) = \sum_{k=1}^{\#\mathcal{T}} r_k \cos(t_k\tau) - r_0, \qquad \mathcal{T} := \{|\boldsymbol{\ell}\cdot\boldsymbol{\zeta}| : \boldsymbol{\zeta} \in \mathfrak{Z}_+^n\}.$$

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Besicovitch mean of
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$$\mathbf{M}[f] := \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s, \qquad (\mathcal{A}[f])(z) := \mathbf{M}\left[\mathrm{e}^{-\mathrm{i}sz} f(s)\right].$$

Then

$$\mathcal{T} = \{z \geq 0 : \mathcal{A}[Q](z) \neq 0\},$$

# Ideas of Proof

Inverse Problem I: from an infinite product to the explicit form of a trigonometric polynomial

Write

$$F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\tau) = \sum_{k=1}^{\#\mathcal{T}} r_k \cos(t_k\tau) - r_0, \qquad \mathcal{T} := \{|\boldsymbol{\ell}\cdot\boldsymbol{\zeta}| : \boldsymbol{\zeta} \in \mathfrak{Z}_+^n\}.$$

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$$\mathcal{T} = \{z \geq 0 : \mathcal{A}[Q](z) \neq 0\}, \quad r_j = 2C\mathcal{A}[Q](t_j),$$

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Then

$$\mathcal{T} = \{z \geq 0 : \mathcal{A}[Q](z) \neq 0\}, \quad r_j = 2C\mathcal{A}[Q](t_j), \quad r_0 = -C\mathcal{A}[Q](0),$$

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with $C$ found from $2C\mathcal{A}[Q](\max\mathcal{T}) = 1$.

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our polynomials are normalised

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Inverse Problem II: $\Sigma \to F$, recover a trigonometric polynomial by its approximate roots

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*Asymptotically isospectral curvilinear polygons have the same quasi-eigenvalues*

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Inverse Problem II: $\Sigma \to F$, recover a trigonometric polynomial by its approximate roots

## Proposition

If $\Sigma$ is the spectrum of a curvilinear polygon $\mathcal{P}(\boldsymbol{\alpha}, \boldsymbol{\ell})$ then

$$F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\tau) = CQ_{\Sigma}(\tau) + o(1) \qquad \text{as } \tau \to +\infty.$$

# Ideas of Proof

Inverse Problem II: $\Sigma \to F$, recover a trigonometric polynomial by its approximate roots

$Q_\Sigma(\tau) := \tau^{2n_0} \prod_{\sigma_j \in \Sigma \setminus \{0\}} \left(1 - \frac{\tau^2}{\sigma_j^2}\right)$

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## Remarks

- Our statement requires a qualified convergence $\sigma_m - \tau_m = O(m^{-\epsilon})$ as $m \to \infty$ rather than $o(1)$.

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- Proof is based on a technical bound $\lim\limits_{\tau\to\infty} \left(Q_\Sigma(\tau) - C_0 Q_{\mathrm{QE}}(\tau)\right) = 0$ with some constant $C_0$.
- Allows the recovery of the frequencies and amplitudes of $F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\tau)$ as before since $\mathcal{A}[f + o(1)](z) = \mathcal{A}[f](z)$ for all $z$.

# Ideas of Proof

Inverse Problem III: $F \to \ell, \pm\mathbf{c}_{\boldsymbol{\alpha}}$, recover geometric information from a trigonometric polynomial

At this step, we need our admissibility conditions.

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$\boldsymbol{\ell}$ incommensurable over $\{-1, 1, 0\}$; no special angles

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immediately gives us the number of vertices $n$

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We will first find $\boldsymbol{\ell}'$ — the permutation of the vector of length in order of magnitude, $\ell_1' < \ell_2' < \dots < \ell_n'$.

Easier to show on a concrete example. We will not need $r_k$'s at this stage.

# Ideas of Proof

Inverse Problem III: recover $\ell'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

$$F(\tau) = \sum_{j=1}^{8} ?\cos(t_j\tau) - ? =$$

$$\boxed{t_j \in \mathcal{T} = \{|\ell \cdot \zeta| : \zeta \in \mathfrak{Z}_+^n\}}$$

$$?\cos(1\tau) + ?\cos(3\tau) + ?\cos(5\tau) + ?\cos(9\tau)$$

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Eight terms, so $n = 4$.

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- Look for the maximal frequency $t_8$

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Eight terms, so $n = 4$.

$$L = 23$$

- Look for the maximal frequency $t_8 = 23$

# Ideas of Proof

## Inverse Problem III: recover $\boldsymbol{\ell}'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

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Eight terms, so $n = 4$.

$$L = 23$$

- Look for the next biggest frequency $t_7$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2,$$

- Look for the next biggest frequency $t_7 = 19 = L - 2\ell_1' = 23 - 2 \times 2$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2,$$

- The next biggest frequency is $t_6$

# Ideas of Proof

## Inverse Problem III: recover $\boldsymbol{\ell}'$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3,$$

- The next biggest frequency is $t_6 = 17 = L - 2\ell'_2 = 23 - 2 \times 3$

# Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

$$F(\tau) = \sum_{j=1}^{8} ?\cos(t_j\tau) - ? =$$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3,$$

- Remove all remaining frequencies in which either $\ell'_1$ or $\ell'_2$ or both come with a minus: $13 = 23 - 2 \times 2 - 2 \times 3$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3,$$

- Remove all remaining frequencies in which either $\ell_1'$ or $\ell_2'$ or both come with a minus: $13 = 23 - 2 \times 2 - 2 \times 3$

# Ideas of Proof

## Inverse Problem III: recover $\boldsymbol{\ell}'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

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# Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}'$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3, 7,$$

- The biggest remaining frequency is $t_4 = 9 = L - 2\ell_3' = 23 - 2 \times 7$

# Ideas of Proof

## Inverse Problem III: recover $\boldsymbol{\ell}'$

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$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3, 7,$$

- Remove all remaining frequencies in which any of $\ell'_1$, $\ell'_2$, or $\ell'_3$ comes with a minus

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## Inverse Problem III: recover $\boldsymbol{\ell}'$

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Inverse Problem III: recover $\boldsymbol{\ell}'$

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remaining frequency is $t_1$

# Ideas of Proof

## Inverse Problem III: recover $\boldsymbol{\ell}'$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3, 7, 11)$$

remaining frequency is $t_1 = 1 = L - 2\ell'_4 = 23 - 2 \times 11$

# Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}$ in proper order and $\mathbf{c}_{\boldsymbol{\alpha}}$

Now we can look at the full polynomial

$$
\begin{aligned}
F(\tau) &= \sum_{j=1}^{8} r_j \cos(t_j \tau) - r_0 \\
&= \frac{1}{3}\cos(\tau) - \frac{1}{60}\cos(3\tau) + \frac{1}{8}\cos(5\tau) + \frac{1}{10}\cos(9\tau) \\
&\quad - \frac{2}{15}\cos(13\tau) - \frac{1}{6}\cos(17\tau) + \frac{1}{20}\cos(19\tau) + \cos(23\tau) + \frac{\sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

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Write now the coefficients as a matrix,

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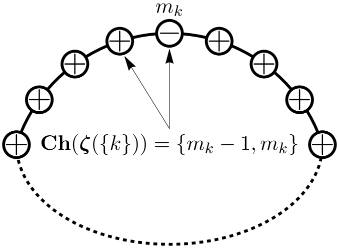


(a)

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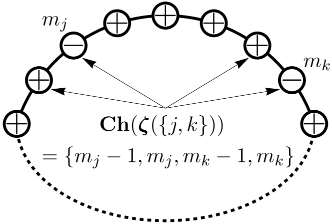


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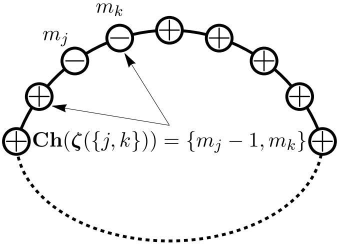

(c)

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*$\ell$ incommensurable over $\{-1, 1, 0\}$; no special angles*

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We may have

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Then $\mathcal{Y}_1 = (\ell'_1)$, $\mathcal{Y}_2 = (\ell'_2)$, $\mathcal{Y}_3 = (\ell'_3, \ell'_4)$.

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## Example 3 — sides commensurable

A pair of curvilinear triangles with sides $\boldsymbol{\ell} = (3, 1, 1)$ and $\tilde{\boldsymbol{\ell}} = (2, 2, 1)$ and cosine vectors $\mathbf{c} = \left(\frac{1}{2}, \frac{1}{2}, \frac{-39+\sqrt{241}}{40}\right)$, $\tilde{\mathbf{c}} = \left(\frac{1}{2}, \frac{7-\sqrt{241}}{12}, \frac{-19+\sqrt{241}}{40}\right)$ are asymptotically isospectral .