Extremal problems on hyperbolic surfaces

GEMSTONE mini-course

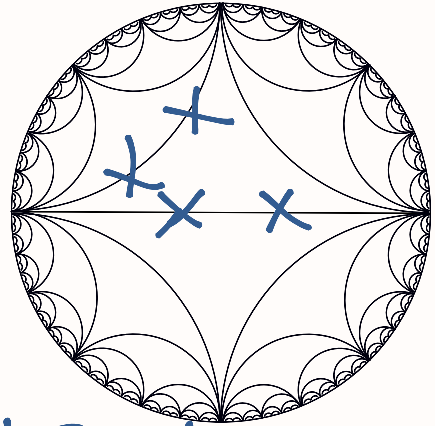
Part I

March 25, 2024

w/ Maxime Fortier Bourque

Hyperbolic surfaces:

Def: A hyperbolic surface is a cplete Riem. 2-mfd of cst curv $\equiv -1$.



$*$ Equivalently: locally isometric to

$$\mathbb{H}^2 = \{x+iy;\ y>0\} \quad ds^2 = \frac{dx^2+dy^2}{y^2}.$$

$$'' = '' \ \{|z|<1\} \quad ds^2 = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$$

Equiv: $\Gamma\backslash\mathbb{H}^2$, $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$

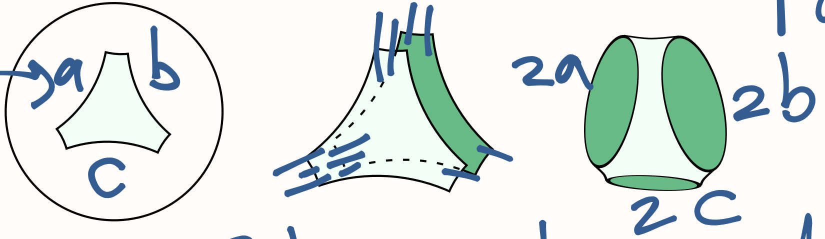
discrete, torsion free

$$\mathrm{Isom}^+(\mathbb{H}^2) = PSl(2,\mathbb{R})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$$

Building hyperbolic surfaces:

Right angled hex:

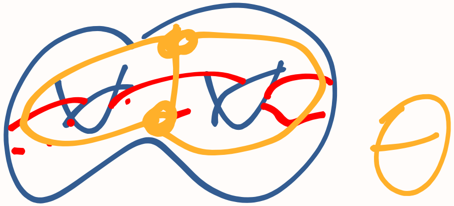


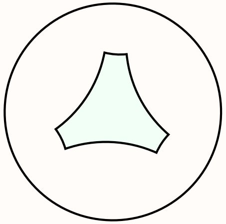
Pair of pants

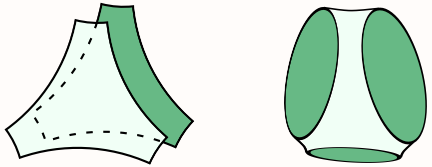
Fact: $*$ $\forall a,b,c$ $\exists!$ (up to isometry) RAH w/ three non-cons. sides of length a,b,c resp.

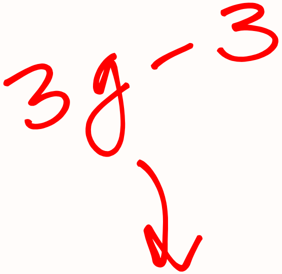
$*$ $\forall a,b,c>0$ $\exists!$ POP w/ bdry comps of length a,b,c resp.

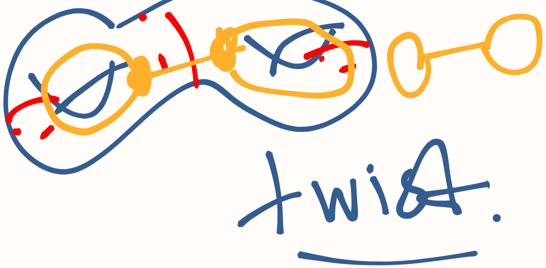
Building hyperbolic surfaces:

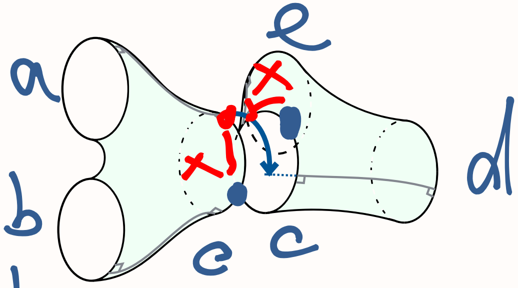


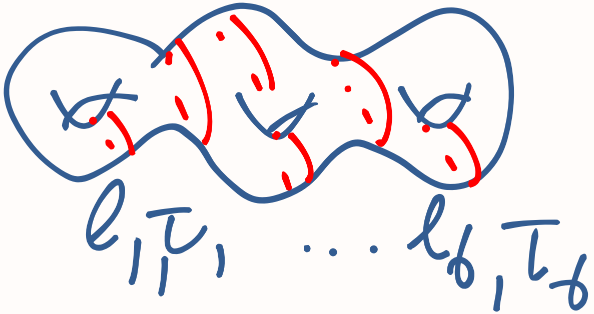












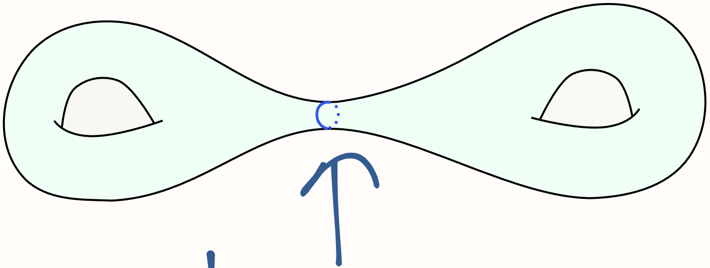
$\mathcal{M}_g = \{$closed hyp. surf. of genus $g\}/$isometry

$(6g-6)$-orbifold

Geometric and spectral invariants:

The systole: $X \in \mathcal{M}_g$

Def: * $\mathrm{Systole}(X) =$ length of the shortest closed geod.



* $\mathrm{Kiss}(X) = \#$ geod. realizing the systole

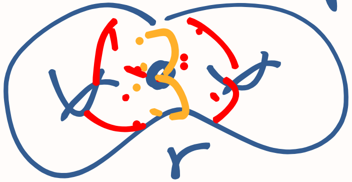
Facts: * $\mathrm{sys} : \mathcal{M}_g \longrightarrow (0,\infty)$ is proper (Mumford)

* Akrout: sys is a top Morse f'n.

<u>Lemma</u> $X \in \mathcal{M}_g$.

$$\mathrm{sys}(X) \le 4\,\mathrm{arcsinh}\left(\sqrt{g-1}\right) \underset{g\to\infty}{=} 2\log(g) + 2.68\ldots + o(1).$$

<u>Proof:</u>



$p \in X$

$$\underset{\pi e^{\mathrm{sys}(X)/2}}{\underset{\wr\wr}{\mathrm{area}\left(D_{\mathrm{sys}(X)/2}(p)\right)}} \le \underset{=4\pi(g-1)}{\underset{\text{Gauß-Bonnet}}{\mathrm{area}(X)}}$$

$\square$

The spectral gap:

$\Delta = -\operatorname{div}\circ\operatorname{grad} : \mathcal{C}^\infty(X) \circlearrowleft$

in $\underset{\mathbb{H}^2}{\underset{\parallel}{\{x+iy;\, y>0\}}}$. : $-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) : \mathcal{C}^\infty(X) \circlearrowleft$

Spectral Thm: Δ has discrete spectrum $0=\lambda_0(X)<\boxed{\lambda_1(X)}\leq\lambda_2(X)\leq\ldots$ and the corresp. eigenfunctions form an ONB for $L^2(X)$.

$m_1(X)$: multiplicity of $\lambda_1(X)$.

Extremal problems on hyperbolic surfaces

Question: Let $g \geq 2$: what are the maxima of systole(X), kiss(X), $\lambda_1(X)$, $m_1(X)$, for $X \in \mathcal{M}_g$?

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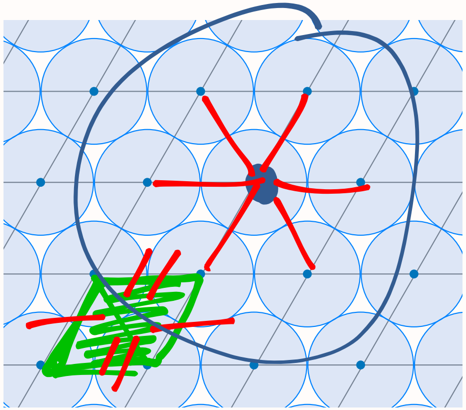
An analogy with packing problems in $\mathbb{R}^n$:

$\Lambda < \mathbb{R}^n$ lattice, i.e.

$\Lambda = \mathrm{span}_{\mathbb{Z}} \{v_1, \dots, v_n\}$

basis



Get a flat torus $\mathbb{R}^n/\Lambda$.

Obs: $\mathrm{systole}(\mathbb{R}^n/\Lambda) = \min_{v \in \Lambda - \{0\}} \|v\|$

Q'n: $\max\{\mathrm{sys}(\mathbb{R}^n/\Lambda) ; \Lambda \text{ of covol } 1\}$?

kissing # = kissing #

Spectral inv:

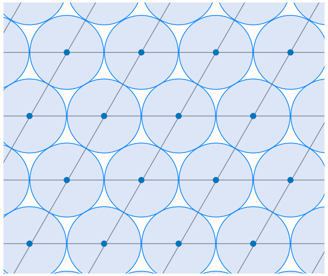
$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Eigenfunctions: $\varphi_w(x) = \exp(2\pi i \langle x, w\rangle)$

$w \in \Lambda^* = \{ w \in \mathbb{R}^n ; \langle v, w\rangle \in \mathbb{Z}, \forall v \in \Lambda \}$

Eigenvalue: $\lambda_w = 4\pi^2 \|w\|^2.$

$$\mathrm{vol}(\mathbb{R}^n/\Lambda) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda^*)}.$$



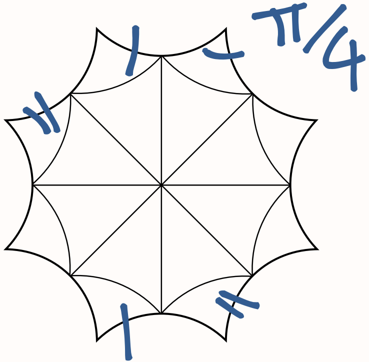
dim.	Density	Kissing number
1	Trivial	Trivial
2	**[Thue, Fejes-Tóth]**	Exercise
3	**[Hales]** FCC	**[Newton,Gregory,...,Schütte–van der Waerden]**
4	?	**[Musin]**
8	**[Viazovska]** E8	**[Levenshteĭn,Odlyzko–Sloane]**
24	**[Cohn–Kumar–Miller–Radchenko–Viazovska]** Leech	**[Levenshteĭn,Odlyzko–Sloane]** 196560

For lattices: Answers known in dimensions 1-8 and 24 for density and 1-9 and 24 for the kissing number.

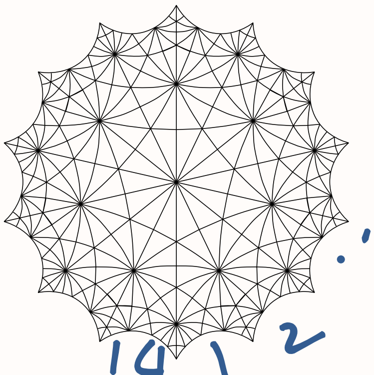
EXTREMAL PROBLEMS ON HYPERBOLIC SURFACES

Question: Let $g \geq 2$: what are the maxima of systole(X), kiss(X), $\lambda_1(X)$, $m_1(X)$, for $X \in \mathcal{M}_g$?

Lots of previous work: Huber '74, Cheng '75, Huber '76, Buser '77, Huber '80, Yang–Yau '80, Jenni '84, Burger Colbois '85, Brooks '88, Burger–Buser–Dodziuk '88, Colbois–Colin-de-Verdière '88, Burger '90, Schmutz '93, Schmutz '94, Buser–Sarnak '94, Bavard '96, Bavard '97, Schmutz-Schaller '97, Adams '98, Hamendstädt '01, Hamenstädt–Koch '02, Kim–Sarnak '03, Casamayou-Boucau '05, Katz–Schaps–Vishne '07, Otal '08, Gendulphe '09, Otal–Rosas '09, Parlier '13, Strohmaier–Uski '13, Fanoni–Parlier '15, Gendulphe '15, Cook '18, Petri–Walker '18, Petri '18, Hide–Magee '21, Jammes '21, Bonifacio '21, Kravchuk–Mazac–Pal '21, Wu–Xue '21, Lipnowski–Wright '21, Fortier Bourque–Rafi '22, Magee–Naud–Puder '22, Anantharaman–Monk '23, and many others.



(a) The Bolza surface



(b) The Klein quartic

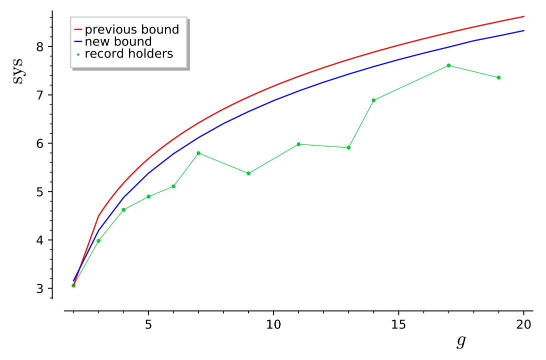
Known maximizers:

	Systole	**Kissing number**	λ_1	$\mathbf{m_1}$
genus 2	Bolza surface **[Jenni ‘84]**	Bolza surface 24, **[Schmutz ’94]**	Conjecture: Bolza surface	Conjecture: Bolza surface
genus 3	Conjecture: Picard curve	Conjecture: Picard curve	Conjecture: Klein quartic	Klein quartic **[Fortier Bourque –P. ‘24+]**
higher genus	Local maximizers **[Schmutz ’99]** **[Hamenstädt ‘01]** **[Fortier Bourque–Rafi ‘22]**			

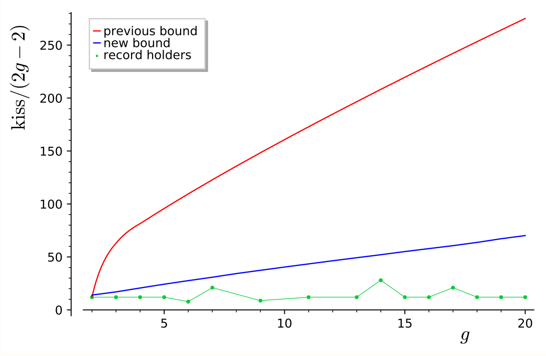
The kissing number in genus 2:

The maximal multiplicity of a Riemannian metric on the 2-sphere [Cheng ’75, Sévennec ’02]:

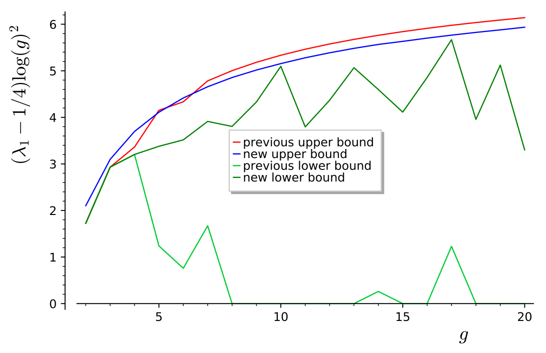
New bounds [Fortier Bourque – P. '23, Fortier Bourque–Gruda-Mediavilla–P.–Pineault '23]



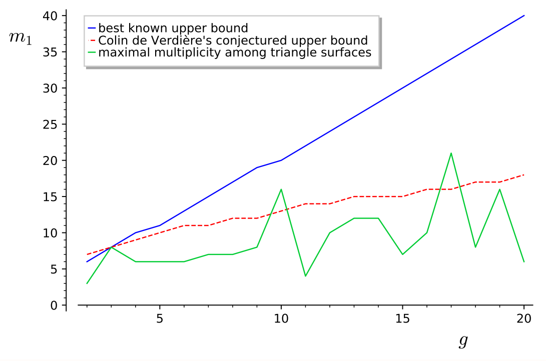
$g = 2$: **[Jenni '84]**



$g = 2$: **[Schmutz '94]**



$g = 2, 3$: **[Bonifacio '21]**, **[Kravchuk–Mazac–Pal '21]**,
$4, 6$: **[Yang–Yau '80]**



Theorem (Fortier Bourque–P. '23) There exists a $g_0 \geq 2$ such that for every hyperbolic surface X of genus $g \geq g_0$:

$$\text{systole}(X) < 2\log(g) + 2.409,$$

$$\text{kiss}(X) < \frac{4.873 \cdot g^2}{\log(g) + 1.2045},$$

$$\lambda_1(X) < \frac{1}{4} + \left(\frac{\pi}{\log(g) + 0.7436}\right)^2,$$

and

$$m_1(X) \leq 2g - 1$$

Best bounds known before: **Bavard '96**, **Fortier Bourque–P. '22** (previously **Parlier '13**), **Cheng '75**, **Sévennec '02** and **Huber '76** respectively.

Sublinear bound on m_1 under the assumption that the systole does not tend to 0 [**Letrouit–Machado '23**]