Lecture 2

Trace formula methods

Plan

① Poisson Summation Formula

② Selberg Trace Formula

③ Upper bounds after Gorbachev - Cohn - Elkies

④ Lower bounds after Booker - Strömbergsson

Def^n A trace formula is an equation that expresses the trace of an operator in two different ways.

Ex $\sum$ diagonal entries $=$ $\operatorname{tr} A$ $= \sum$ eigenvalues

More loosely, it is an equation that relates the eigenvalues of an operator to other data.

Poisson Summation Formula

If $f: \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function, $\Lambda \leq \mathbb{R}^n$ is a lattice, and $\Lambda^* = \{w \in \mathbb{R}^n : \langle v, w \rangle \in \mathbb{Z} \ \forall v \in \Lambda\}$, then

$$\sum_{v \in \Lambda} f(v) = \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \sum_{w \in \Lambda^*} \hat{f}(2\pi w)$$

$\boxed{\text{This is a trace formula for } \Delta \text{ on } \mathbb{R}^n/\Lambda\text{!}}$

Def^n $f \in \mathcal{S}(\mathbb{R}^n)$ if all partial derivatives of f (of all orders) decay faster than $\frac{1}{\|x\|^k}$ as $\|x\| \to \infty$, $\forall k \in \mathbb{N}$.

Pf of PSF Define $F(x) := \sum_{v \in \Lambda} f(x+v)$. $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow F \in C^\infty$ and by construction, F is Λ-periodic.

$$\Longrightarrow \quad F = \sum_{w \in \Lambda^*} c_w e_w \quad \text{in } L^2 \text{ for some } c_w \in \mathbb{C}$$

where $e_w(x) = e^{2\pi i \langle x, w \rangle}$. These are eigenfunctions of $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ on $X = \mathbb{R}^n / \Lambda$ with eigenvalue $(2\pi \|w\|)^2$.

We compute

$$c_w \operatorname{vol}(X) = c_w \|e_w\|^2 = \langle F, e_w \rangle$$

$$= \int_X \sum_{v \in \Lambda} f(x+v)\, e^{-2\pi i \langle x, w \rangle}\, dx$$

P fund. domain for Λ.

$$= \sum_{v \in \Lambda} \int_P f(x+v)\, e^{-2\pi i \langle x+v, w \rangle}\, d(x+v)$$

$$= \sum_{v \in \Lambda} \int_{P+v} f(u)\, e^{-2\pi i \langle u, w \rangle}\, du$$

$$= \int_{\mathbb{R}^n} f(u) e^{-i\langle u, 2\pi w \rangle} du = \hat{f}(2\pi w)$$

$$\Rightarrow \quad \sum_{v\in\Lambda} f(x+v) = F(x) = \frac{1}{\mathrm{Vol}(X)} \sum_{w\in\Lambda^*} \hat{f}(2\pi w)\, e^{2\pi i\langle x, w\rangle} \quad \text{in } L^2$$

and hence pointwise since both sides are continuous.

Evaluating at $x=0$ gives the PSF. $\square$

<u>Why is it a trace formula?</u>

Suppose $f(x) = \phi(\|x\|)$. Then $\hat{f}(y) = \psi(\|y\|)$.

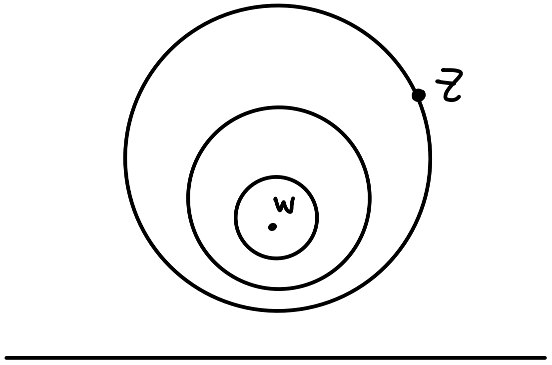
$$\phi(0) + \sum \phi(\ell(\gamma)) = \sum_{v\in\Lambda} \phi(\|v\|) = \frac{1}{\mathrm{Vol}(X)} \sum_{\lambda\in\mathrm{spec}(\Delta_X)} \psi(\sqrt{\lambda})$$

homotopy classes of oriented closed geodesics in X

② Towards the Selberg Trace Formula

We want to do the same thing if $X = \mathbb{H}^2/\Gamma$ is a closed hyperbolic surface.

Start with a radial function $k(z,w) = K(d(z,w))$



Define $\underline{K}(z) := \sum_{\gamma \in \Gamma} k(z, \gamma z)$.

$\forall \alpha \in \Gamma$, we have

$$K(\alpha z) = \sum_{\gamma \in \Gamma} k(\alpha z, \gamma \alpha z) = \sum_{\gamma \in \Gamma} k(z, \alpha^{-1}\gamma\alpha z) = \sum_{\sigma \in \Gamma} k(z, \sigma z) = K(z)$$

$$\Rightarrow \quad K = \sum_{\lambda \in \mathrm{spec}(\Delta_X)} c_\lambda \, \phi_\lambda$$ where ϕ_λ form a basis of eigenfunctions of Δ_X.

Instead of evaluating at a particular point, we integrate over X.

Length-trace formula

$$\int_X K(z)\,dz = k(0)\,\mathrm{area}(X) + \sum_{[\gamma] \in \mathcal{P}(X)} \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sqrt{\cosh(\ell(\gamma^n)) - 1}} \int_{\ell(\gamma^n)}^{\infty} \frac{k(\rho)\sinh\rho \, d\rho}{\sqrt{\cosh\rho - \cosh(\ell(\gamma^n))}}$$

where $\mathcal{P}(X)$ = conjugacy classes of primitive elements in Γ.

<u>Sketch</u> Let D be a fundamental domain for Γ. We have

$$\int_X K(z)\,dz = \int_D \sum_{\gamma\in\Gamma} k(z,\gamma z)\,dz = \sum_{\gamma\in\Gamma} \int_D k(z,\gamma z)\,dz.$$

① If $\gamma = id$ get $\int_D k(z,z)\,dz = \int_D k(0)\,dz = k(0)\,\text{area}(X)$.

② If $\alpha\in\Gamma$, then

$$\int_D k(z,\alpha^{-1}\gamma\alpha z)\,dz = \int_D k(\alpha z,\gamma\alpha z)\,d(\alpha z) = \int_{\alpha D} k(z,\gamma z)\,dz$$

③ $\alpha^{-1}\gamma\alpha = \beta^{-1}\gamma\beta \iff \beta\alpha^{-1}$ commutes with γ

$\iff \beta\alpha^{-1}$ and γ are both powers of a common primitive γ_0

$\iff \gamma = \gamma_0^n$ and $\alpha\langle\gamma_0\rangle = \beta\langle\gamma_0\rangle$

$$\Rightarrow \sum_{\gamma\in\Gamma\setminus\{id\}} \text{ becomes } \sum_{[\gamma_0]\in\mathcal{P}(X)} \sum_{n=1}^{\infty} \sum_{\alpha\langle\gamma_0\rangle\in\Gamma/\langle\gamma_0\rangle} \int_{\alpha D} k(z,\gamma_0^n z)\,dz$$

Moreover,

$$\sum_{\alpha\langle\gamma_0\rangle\in\Gamma/\langle\gamma_0\rangle} \int_{\alpha D} k(z,\gamma_0^n z)\,dz = \int_{\mathbb{H}^2/\langle\gamma_0\rangle} k(z,\gamma_0^n z)\,dz.$$

$\hookrightarrow$ can compute explicitly by conjugating γ_0 to

$$\begin{pmatrix} e^{\ell(\gamma_0)/2} & 0 \\ 0 & e^{-\ell(\gamma_0)/2} \end{pmatrix}$$

$$\left(\mathbb{H}^2/\langle\gamma_0\rangle\right)\Big/\left(\Gamma/\langle\gamma_0\rangle\right) \cong \mathbb{H}^2/\Gamma$$

$\square$

The Selberg Trace Formula

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is admissible and $X = \mathbb{H}^2/\Gamma$ is closed, then

$$\sum_{\lambda \in \operatorname{spec}(\Delta_X)} \hat{f}\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = \frac{\operatorname{area}(X)}{2\pi} \int_0^\infty \hat{f}(r)\, r \tanh(\pi r)\, dr$$

Idea: Define K in terms of f via the Abel transform.

$$+ \sum_{\gamma \in P(X)} \sum_{n=1}^{\infty} \frac{\ell(\gamma) f(n\ell(\gamma))}{2\sinh\left(n\ell(\gamma)/2\right)}.$$

Def^n $f : \mathbb{R} \to \mathbb{R}$ is admissible if it is even, $\hat{f}$ is well-defined on $\{|\operatorname{Im} \xi| < \tfrac{1}{2} + \varepsilon\}$ for some $\varepsilon > 0$ and

$$|\hat{f}(\xi)| = O\left(\frac{1}{|\xi|^{2+\varepsilon}}\right) \quad \text{in that strip.}$$

③ Upper bounds from trace formulas

<u>Thm</u> (Gorbachev 2000, Cohn-Elkies 2003)

If $f \in \mathcal{S}(\mathbb{R}^n)$ and $r > 0$ is such that

① $f(x) \leq 0$ if $\|x\| \geq r$

② $\hat{f}(y) \geq 0$ $\forall y \in \mathbb{R}^n$

③ $f(0) = \hat{f}(0) = 1$.

Then $\mathrm{sys}(X) \leq r$ for every flat torus X of dimension n and volume 1.

<u>Pf</u> Let $X = \mathbb{R}^n / \Lambda$. We prove that $\dfrac{\mathrm{sys}(X)^n}{\mathrm{vol}(X)} \leq r^n$, which is equivalent.

Scale X so that $\mathrm{sys}(X) = r$, which does not change the systolic ratio. Then

$$1 = f(0) \geq \sum_{v \in \Lambda} f(v) \overset{\text{PSF}}{=} \frac{1}{\mathrm{vol}(X)} \sum_{w \in \Lambda^*} \hat{f}(2\pi w) \geq \frac{\hat{f}(0)}{\mathrm{vol}(X)} = \frac{1}{\mathrm{vol}(X)}$$

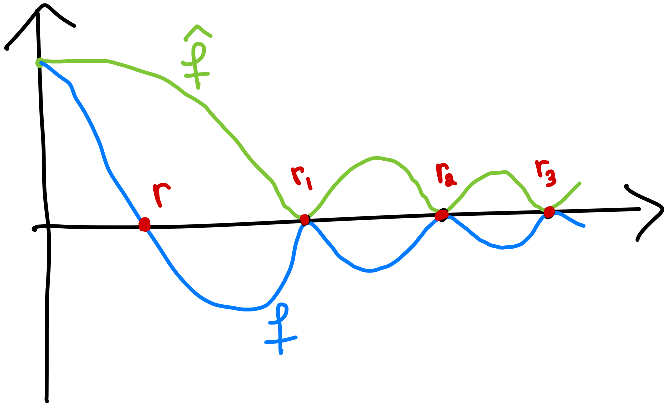
from which the ineq. follows. $\square$

How to optimize the bound?

Take $f(x) = p(\|x\|^2)\, e^{-\|x\|^2/2}$ where p is a polynomial, so that

$$\hat{f}(y) = q(\|y\|^2)\, e^{-\|y\|^2/2}$$ for a polynomial q as well.

Impose some double zeros $r_1, r_2, \ldots, r_m$ to p & q simultaneously, which gives a linear system of equations.



Then vary $(r_1, \ldots, r_m)$ and optimize numerically.

An analogous method works for hyperbolic surfaces!

<u>Observation</u> $\mathrm{Sys}(X) \leq r \iff$ the number of closed geodesics with length in $(0, r]$ is at least 1.

<u>Generalization</u> (FB-Petri)

Let E be a multiset of elements in $\mathbb{R}$ and let $I \subseteq \mathbb{R}$.

Suppose that ① $\Psi(\lambda) \leq 0 \quad \forall \lambda \in E \setminus I$

② $\displaystyle\sum_{\lambda \in E} \Psi(\lambda) > 0$

③ $m \Psi(\mu) < \displaystyle\sum_{\lambda \in E} \Psi(\lambda)$ for all $\mu \in I$.

Then $\# E \cap I \geq m+1$ counting multiplicity.

We can use the trace formula to check ② and ③!

④ Lower bounds on spectral gaps (for specific X's)

Lemma (Booker–Strömbergsson '07) Let X be a hyperbolic surface. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is admissible, $\hat{f}(\sqrt{\lambda-1/4})\geq 0$ $\forall\lambda\geq 0$ and

$$\hat{f}(\sqrt{\mu-1/4}) > \sum_{\lambda\in\operatorname{spec}(\Delta_X)} \hat{f}(\sqrt{\lambda-1/4}).$$

Then $\mu\notin\operatorname{spec}(\Delta_X)$.

Pf If $\mu\in\operatorname{spec}(\Delta_X)$, then $\hat{f}(\sqrt{\mu-1/4})\leq\sum_{\lambda\in\operatorname{spec}(\Delta_X)}\hat{f}(\sqrt{\lambda-1/4})$ since the other terms are ≥ 0. ∎

Goal: Maximize

$$\frac{\hat{f}(\sqrt{\mu-1/4})}{\sum_{\lambda\in\operatorname{spec}(\Delta_X)}\hat{f}(\sqrt{\lambda-1/4})}$$

over all f.

How to optimize over f?
(to rule out a particular μ)

Suppose we know the primitive length spectrum of X up to length L.

Let $h_1, \ldots, h_n : \mathbb{R} \to \mathbb{R}$ be such that $h_j * h_k$ is admissible and supported in $[-L, L]$ for every j, k.

For $x \in \mathbb{R}^n$, let $h = \sum_{j=1}^{n} x_j h_j$ and $f = h * h = \sum_{j,k} x_j x_k h_j * h_k$

so that $\hat{f} = (\hat{h})^2 \geq 0$ on $\mathbb{R} \cup i\mathbb{R}$.

Write $b(t) := (\hat{h}_1(t), \ldots, \hat{h}_n(t))$ so that $\hat{h}(t) = \langle x, b(t) \rangle$ and

$\hat{f}(t) = \langle x, b(t) \rangle^2$.

Define $A_{j,k} := \sum_{\lambda \in \mathrm{spec}(\Delta_X)} \widehat{h_j * h_k}\left(\sqrt{\lambda - 1/4}\right)$ so that

$$\sum_{\lambda \in \mathrm{spec}(\Delta_X)} \hat{f}\left(\sqrt{\lambda - 1/4}\right) = \sum_{\lambda \in \mathrm{spec}(\Delta_X)} \sum_{j,k=1}^{n} x_j x_k \, \widehat{h_j * h_k}\left(\sqrt{\lambda - 1/4}\right)$$

$$= \sum_{j,k=1}^{n} x_j A_{j,k} x_k = \langle x, Ax \rangle$$

We now want to find

$$\max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, b \rangle^2}{\langle x, Ax \rangle}$$

where $b = b\left(\sqrt{\mu - 1/4}\right)$

(if > 1, then $\mu \notin \mathrm{spec}(\Delta_X)$)

A is symmetric and positive definite, so it admits a symmetric positive definite square root $\sqrt{A}$.

$$\Rightarrow \langle x, Ax \rangle = \langle \sqrt{A}x, \sqrt{A}x \rangle \quad \text{so set } y = \sqrt{A}x$$

$$\Rightarrow x = (\sqrt{A})^{-1} y$$

and $\langle x, b \rangle = \langle y, (\sqrt{A})^{-1} b \rangle$

By Cauchy-Schwarz, $\dfrac{\langle y, (\sqrt{A})^{-1} b \rangle^2}{\|y\|^2} \leq \|(\sqrt{A})^{-1} b\|^2$ with equality only if $y \propto (\sqrt{A})^{-1} b \iff x \propto A^{-1} b$.

The Booker-Strömbergsson function for the Bolza surface with $L=9.1$

$n=26$

$$BS(\mu) := \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, Ax \rangle}{\langle x, b(\sqrt{\mu - 1/4}) \rangle^2}$$