Extremal problems on hyperbolic surfaces

GEMSTONE mini-course

Part III

March 25, 2024

Recap:

$$\mathcal{M}_g = \left\{ \begin{array}{c} \text{closed orientable hyperbolic} \\ \text{surfaces of genus } g \end{array} \right\} \Big/ \text{isometry}, \quad g \geq 2.$$

Recap:

$$\mathcal{M}_g = \left\{ \begin{array}{c} \text{closed orientable hyperbolic} \\ \text{surfaces of genus } g \end{array} \right\} \Big/ \text{isometry}, \quad g \geq 2.$$

For $X \in \mathcal{M}_g$,

- systole(X) is the length of the shortest closed geodesic on X,
- Kiss(X) is the number of oriented closed geodesics realizing systole(X),
- $\lambda_1(X)$ is the smallest non-zero eigenvalue of the Laplacian $\Delta : C^\infty(X) \to C^\infty(X)$ and
- $m_1(X)$ is the multiplicity of $\lambda_1(X)$.

Recap:

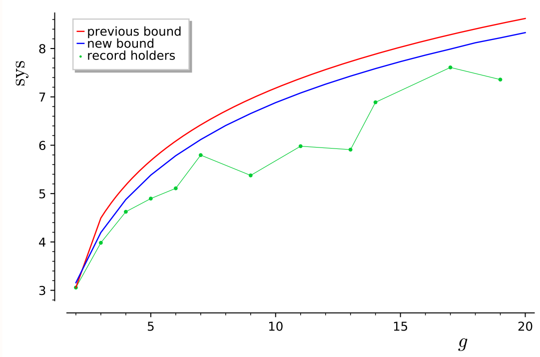
$$\mathcal{M}_g = \left\{ \begin{array}{c} \text{closed orientable hyperbolic} \\ \text{surfaces of genus } g \end{array} \right\} \Big/ \text{ isometry}, \quad g \geq 2.$$

For $X \in \mathcal{M}_g$,

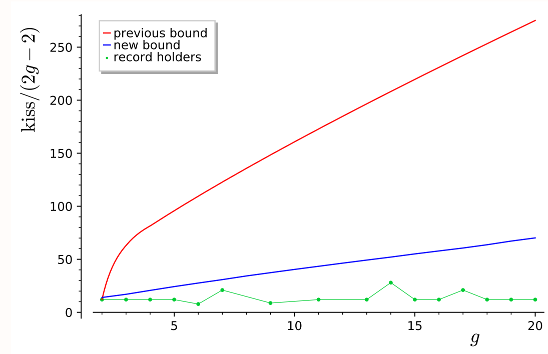
- systole(X) is the length of the shortest closed geodesic on X,
- Kiss(X) is the number of oriented closed geodesics realizing systole(X),
- $\lambda_1(X)$ is the smallest non-zero eigenvalue of the Laplacian $\Delta : C^\infty(X) \to C^\infty(X)$ and
- $m_1(X)$ is the multiplicity of $\lambda_1(X)$.

Question: Let $g \geq 2$: what are the maxima of systole(X), kiss(X), $\lambda_1(X)$, $m_1(X)$, for $X \in \mathcal{M}_g$?

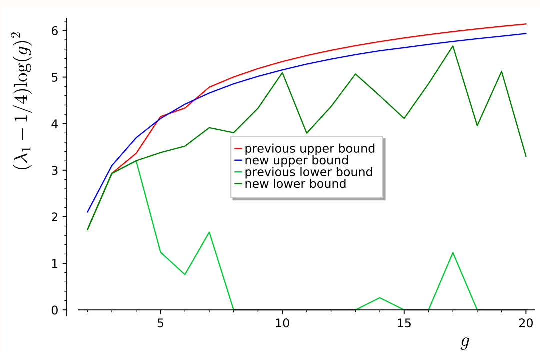
New bounds [Fortier Bourque – P. ‘23, Fortier Bourque–Gruda-Mediavilla–P.–Pineault ’23]



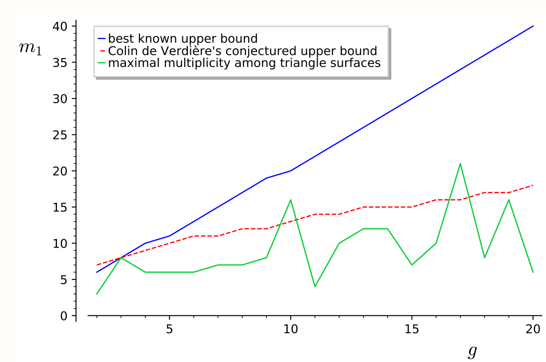
$g = 2$: **[Jenni ’84]**



$g = 2$: **[Schmutz ’94]**



$g = 2, 3$: **[Bonifacio ’21]**, **[Kravchuk–Mazac–Pal ’21]**, $4, 6$: **[Yang–Yau ’80]**



Theorem (Fortier Bourque–P. ’23) There exists a $g_0 \geq 2$ such that for every hyperbolic surface X of genus $g \geq g_0$:

$$\text{systole}(X) < 2\log(g) + 2.409,$$

$$\text{kiss}(X) < \frac{4.873 \cdot g^2}{\log(g) + 1.2045},$$

$$\lambda_1(X) < \frac{1}{4} + \left(\frac{\pi}{\log(g) + 0.7436}\right)^2,$$

and

$$m_1(X) \leq 2g - 1$$

Best bounds known before: **Bavard ’96**, **Fortier Bourque–P. ’22** (previously **Parlier ’13**), **Cheng ’75**, **Sévennec ’02** and **Huber ’76** respectively.

Sublinear bound on m_1 under the assumption that the systole does not tend to 0 [**Letrouit–Machado ’23**]

$\forall \varepsilon > 0 \ \exists C_\varepsilon > 0$ s.t. $m_1(X) \leq \dfrac{C_\varepsilon \cdot g}{\log\log(g)}$ $\forall X \in \mathcal{M}_g$ w sys $> \varepsilon$.

Bounds based on trace formulas:

The Selberg trace formula: $f : \mathbb{R} \to \mathbb{R}$ admissible, $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-ix \cdot \xi)\, dx$, $X \in \mathcal{M}_g$, then

$$\sum_{n \geq 0} \widehat{f}\left(\sqrt{\lambda_n - \frac{1}{4}}\right) = 2(g-1) \int_0^\infty \widehat{f}(y) \tanh(\pi y) y \, dy + \sum_{\substack{\gamma \text{ prim. closed} \\ \text{geod. on } X}} \ell(\gamma) \sum_{k \geq 1} \frac{f(k \cdot \ell(\gamma))}{2 \sinh(k \cdot \ell(\gamma)/2)}$$

Bounds based on trace formulas:

The Selberg trace formula: $f : \mathbb{R} \to \mathbb{R}$ admissible, $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-ix \cdot \xi)\, dx$, $X \in \mathcal{M}_g$, then

$$\sum_{n \geq 0} \widehat{f}\left(\sqrt{\lambda_n - \frac{1}{4}}\right) = 2(g-1) \int_0^\infty \widehat{f}(y) \tanh(\pi y) y \, dy + \sum_{\substack{\gamma \text{ prim. closed} \\ \text{geod. on } X}} \ell(\gamma) \sum_{k \geq 1} \frac{f(k \cdot \ell(\gamma))}{2 \sinh(k \cdot \ell(\gamma)/2)}$$

Linear programming bound:

Let $g \geq 2$. Suppose that f is a non-constant admissible function and $L > 0$ such that:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\widehat{f}\left(\sqrt{\lambda - \frac{1}{4}}\right) \leq 0$ whenever $\lambda \geq L$
- $\widehat{f}(i/2) \leq 2(g-1) \int_0^\infty \widehat{f}(y) \tanh(\pi y) y \, dy$

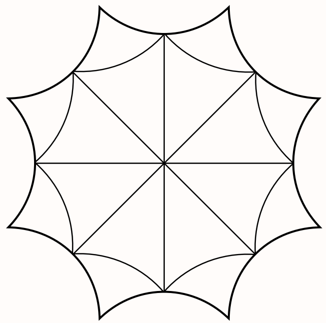
Then $\lambda_1(X) \leq L$ for all $X \in \mathcal{M}_g$.

kth Laguerre poly

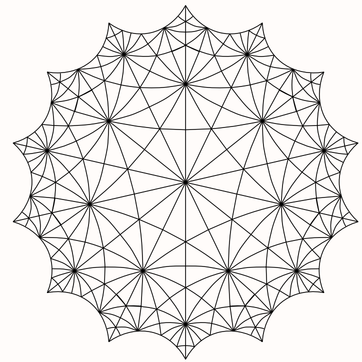
$f(x) = p_k(x^2) e^{-x^2/2}$

$\hat{f}(\xi) = (-1)^k f(\xi)$

Symmetry:



(a) The Bolza surface



(b) The Klein quartic

Set-up: $X = \mathbb{H}^2 \big/ \text{by isometries} \in \mathcal{M}_g \qquad G \circlearrowright X$

Thm (Hurwitz) $\#G \leq 168(g-1)$

and if all or. pres $\#G \leq 84(g-1)$.

Then $G \curvearrowright C^\infty(X)$ by

$$(gf)(x) = f(g^{-1}x) \qquad g \in G, \ f \in C^\infty, \ x \in X$$

and

$$\Delta gf = g \Delta f$$

In part. $E_\lambda = \{ f \in C^\infty(X) ; \Delta f = \lambda f \}$ admits a G-action

$f \in E_\lambda$ then $\Delta gf = g \Delta f = g \cdot \lambda f$

$$= \lambda (gf).$$

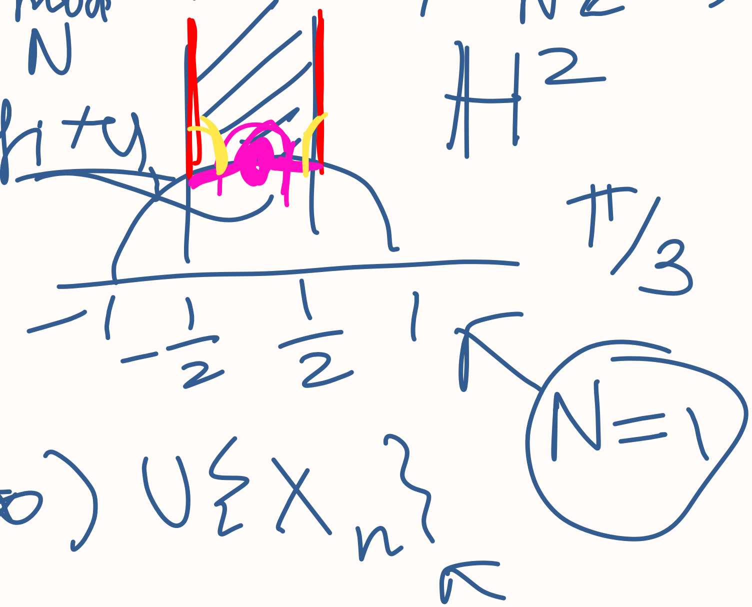
Example 1: High multiplicity for small eigenvalues [Sarnak–Xue '91]

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{Z}) : \begin{array}{l} a \equiv d \equiv 1 \ (N) \\ b \equiv c \equiv 0 \ (N) \end{array} \right\}$$

$$= \ker\left(PSL(2,\mathbb{Z}) \xrightarrow{\text{mod } N} PSL(2, \mathbb{Z}/N\mathbb{Z}) \right)$$

$$X(N) = \Gamma(N) \backslash \mathbb{H}^2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ fixes } i$$



$$PSL(2, \mathbb{Z}/N\mathbb{Z}) \curvearrowright X(N)$$

$$\sigma(\Delta_{X(N)}) = \{0\} \cup [\tfrac{1}{4}, \infty) \cup \{\lambda_n\}$$

<u>Fact</u>: $PSL(2, \mathbb{Z}/p\mathbb{Z})$ has no non-triv irreps of dim $< p-1$

Suppose λ e.v. of Δ $X(p)$

$\Rightarrow E_\lambda = \bigoplus (V^{triv})^m \bigoplus V^\rho$

ρ non-triv

Λ

$f \in A$

$\Rightarrow gf = f \quad \forall g \in PSL(2, p)$

$\Rightarrow f$ descends to $X(p)/PSL(2,p) = \mathbb{H}^2 / PSL(2,\mathbb{Z}) = X(1)$

Bocher-Strömbergsson-Venkatesh

$\lambda_1(X(1)) = 91.14\ldots$

Example 2: Twisted Laplacians and a decomposition of the spectrum

Proposition: If $\Lambda < \Gamma$ are co-compact Fuchsian groups and $G = \Gamma/\Lambda$ is finite, then

$$\operatorname{spec}(\Lambda\backslash\mathbb{H}^2) = \bigcup_{\phi \in \operatorname{Irr}(G)} \dim(\phi) \cdot \operatorname{spec}(\Gamma\backslash\mathbb{H}^2, \phi)$$

as multisets.

$X = \Lambda\backslash\mathbb{H}^2$ $\quad G < \operatorname{Isom}(X)$

X/G hyperbolic orbifold $\Gamma\backslash\mathbb{H}^2$

$\Gamma < PSL(2,\mathbb{R})$ discrete

Example 2: Twisted Laplacians and a decomposition of the spectrum

Proposition: If $\Lambda < \Gamma$ are co-compact Fuchsian groups and $G = \Gamma/\Lambda$ is finite, then

$$\operatorname{spec}(\Lambda\backslash\mathbb{H}^2) = \bigcup_{\phi\in\operatorname{Irr}(G)} \dim(\phi)\cdot\operatorname{spec}(\Gamma\backslash\mathbb{H}^2,\phi)$$

as multisets.

The twisted Selberg trace formula:

$$\sum_{\lambda\in\operatorname{spec}(\Gamma\backslash\mathbb{H}^2,\varphi)} \widehat{f}\left(\sqrt{\lambda-\frac{1}{4}}\right) = \dim(\varphi)\frac{\operatorname{area}(\Gamma\backslash\mathbb{H}^2)}{4\pi}\int_{-\infty}^{\infty} y\widehat{f}(y)\tanh(\pi y)\,dy$$
$$+\sum_{[\gamma]\in\mathcal{E}(\Gamma)}\frac{\operatorname{tr}(\varphi(\gamma))}{2m(\gamma)\sin(\theta(\gamma))}\int_{-\infty}^{\infty}\frac{e^{-2\theta(\gamma)y}}{1+e^{-2\pi y}}\widehat{f}(y)\,dy$$
$$+\sum_{[\gamma]\in\mathcal{P}(\Gamma)}\ell(\gamma)\sum_{n\geq 1}\frac{\operatorname{tr}(\varphi(\gamma^n))}{2\sinh(n\ell(\gamma)/2)}f(n\ell(\gamma))$$

Twisted Laplacians:

$$C^\infty(\Gamma\backslash\mathbb{H}^2) \simeq C^\infty(\mathbb{H}^2)^\Gamma \qquad \phi \in \mathrm{Irr}(G)$$

$$C^\infty(\Gamma\backslash\mathbb{H}^2, \phi) = \{F : \mathbb{H}^2 \to V ;\ F(\gamma x) = \phi(\gamma) F(x)\}$$

$$\Delta_\phi : C^\infty(\Gamma\backslash\mathbb{H}^2, \phi) \circlearrowleft \qquad \phi : G \to GL(V)$$

coord. wise

$$\Lambda = \ker(\Gamma \to G)$$

$$\mathrm{spec}(\Gamma\backslash\mathbb{H}^2, \phi) = \mathrm{spec}(\Delta_\phi) \text{ discrete}$$

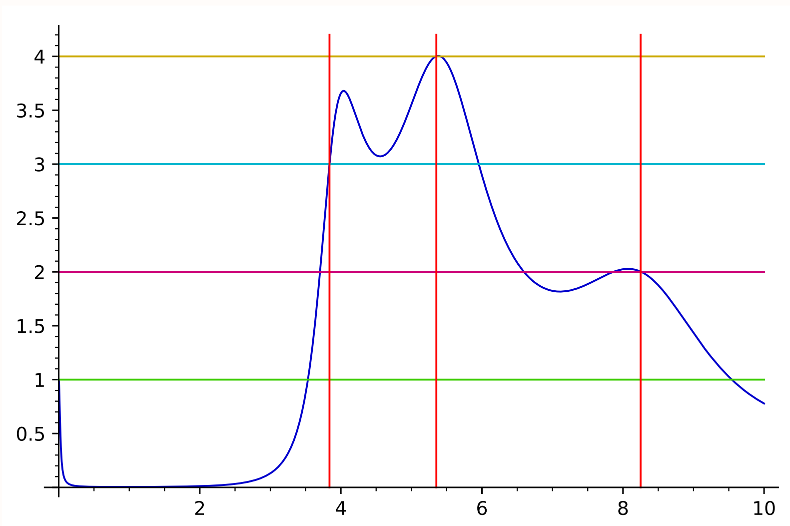
The group algebra:

Proposition: If $\Lambda < \Gamma$ are co-compact Fuchsian groups and $G = \Gamma/\Lambda$ is finite, then

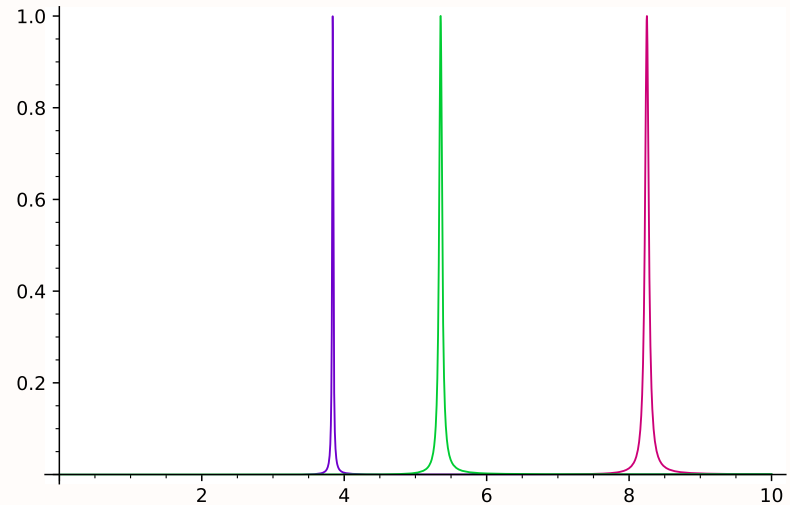
$$\operatorname{spec}(\Lambda\backslash\mathbb{H}^2) = \bigcup_{\phi\in\operatorname{Irr}(G)} \dim(\phi)\cdot\operatorname{spec}(\Gamma\backslash\mathbb{H}^2,\phi)$$

as multisets.

Two plots for the spectrum of the Bolza surface:



Not using the representations ($L = 9.1$, $n = 26$)



Using the representations ($L = 7$, $n = 6$)
Degrees: $3, 4, 2$

$\mathrm{Isom}^+(B) \simeq \mathrm{GL}(2, \mathbb{Z}/3\mathbb{Z})$. Character degrees over $\mathbb{R}$:

$$1, 1, 2, 4, 3, 3, 4$$

Theorem [Fortier Bourque – P. '21]: We have

$$\max_{X \in \mathcal{M}_3} \{m_1(X)\} = 8$$

and this is realized by the Klein quartic

Open question: Is the Klein quartic the unique surface in $\mathcal{M}_3$ with $m_1 = 8$?

If $\lambda_1(X(p)) \leq g!$

then $m_1(X(p)) \geq p-1$

$\approx \text{area}(X(p))^{1/3}$