Lecture 4

Triangle surfaces

① Triangle groups and surfaces

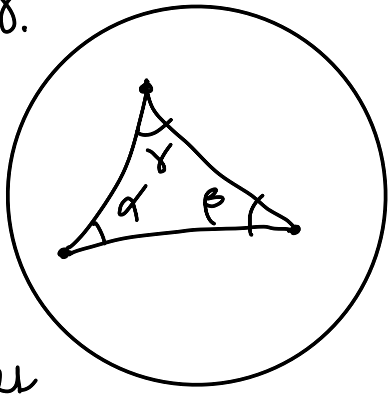
② Spectral data

③ Representation theory & multiplicity

①

Triangle groups

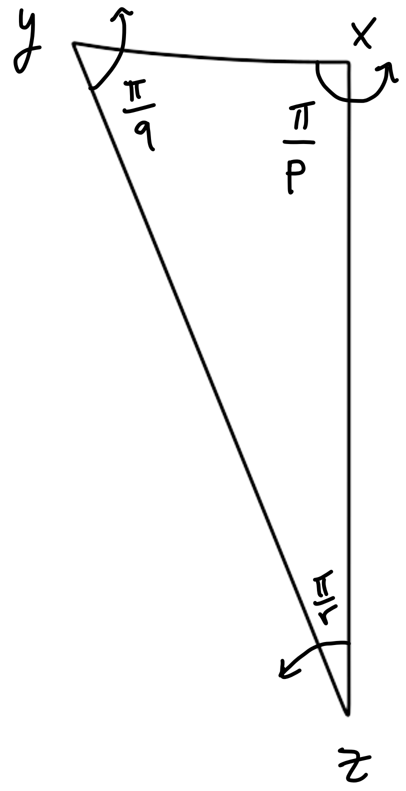
Fact $\forall \alpha, \beta, \gamma > 0$ s.t. $\alpha+\beta+\gamma<\pi$, there exists a (unique up to isom) hyperbolic triangle with interior angles α, β, γ.



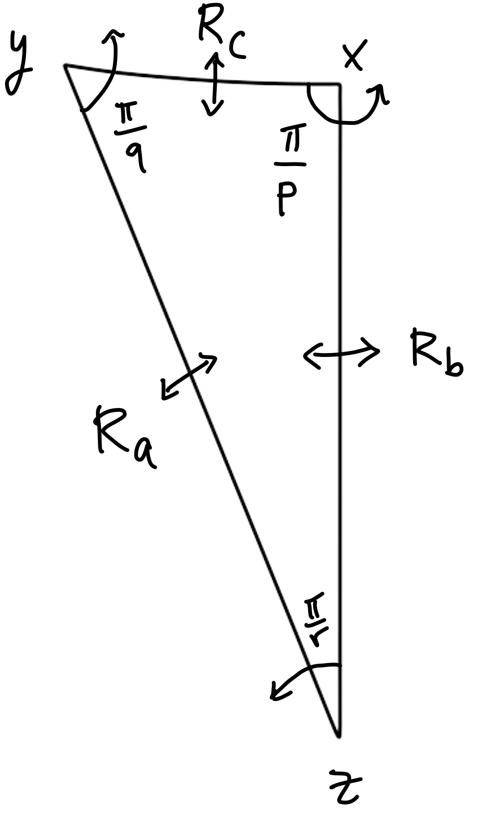
Def^n
- $(p,q,r)\in\mathbb{N}^3$ is hyperbolic if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$.
- A (p,q,r)-triangle is a triangle with interior angles π/p, π/q, π/r. (exists by the fact)
- The (p,q,r)-triangle group $T(p,q,r)$ is the group generated by the counterclockwise rotations of angle $2\pi/p$, $2\pi/q$, $2\pi/r$ about the vertices of a (p,q,r)-triangle. (unique up to conj)

Relations

We have $x^p = y^q = z^r = id$



Relations



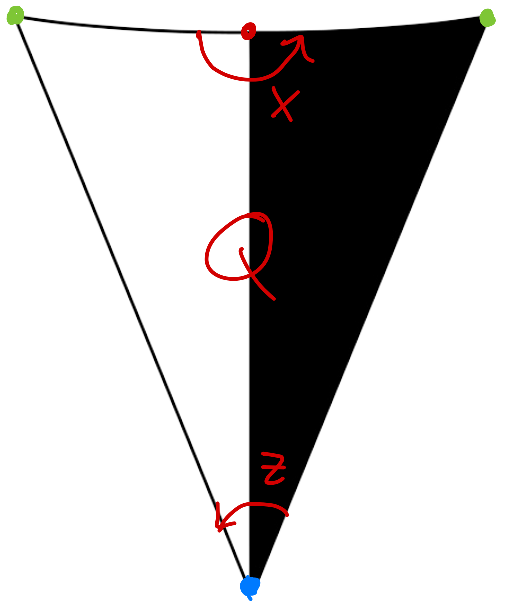
We have $x^p = y^q = z^r = id$

We can write
$$x = R_b R_c$$
$$y = R_c R_a$$
$$z = R_a R_b$$

$$\Rightarrow \quad xyz = R_b R_c R_c R_a R_a R_b$$
$$= R_b R_b = id.$$

Cor T is generated by any two of x, y, z.

Fundamental domain



By the Poincaré polygon theorem, Q is a fundamental domain for the action of $T = T(p,q,r)$ on $\mathbb{H}^2$.

In other words,

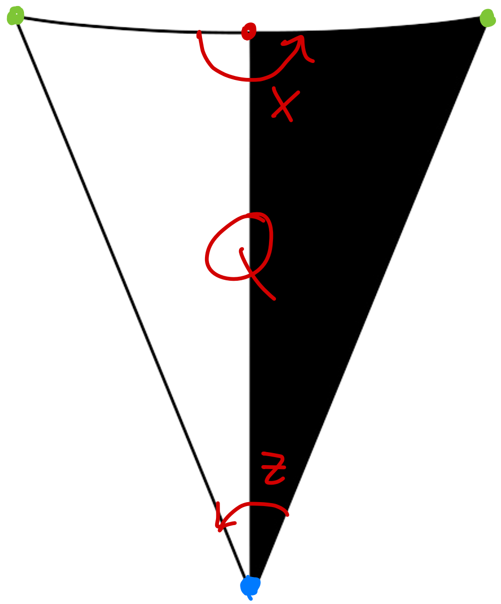
$$\{\gamma(Q) : \gamma \in T\} \text{ tile } \mathbb{H}^2.$$

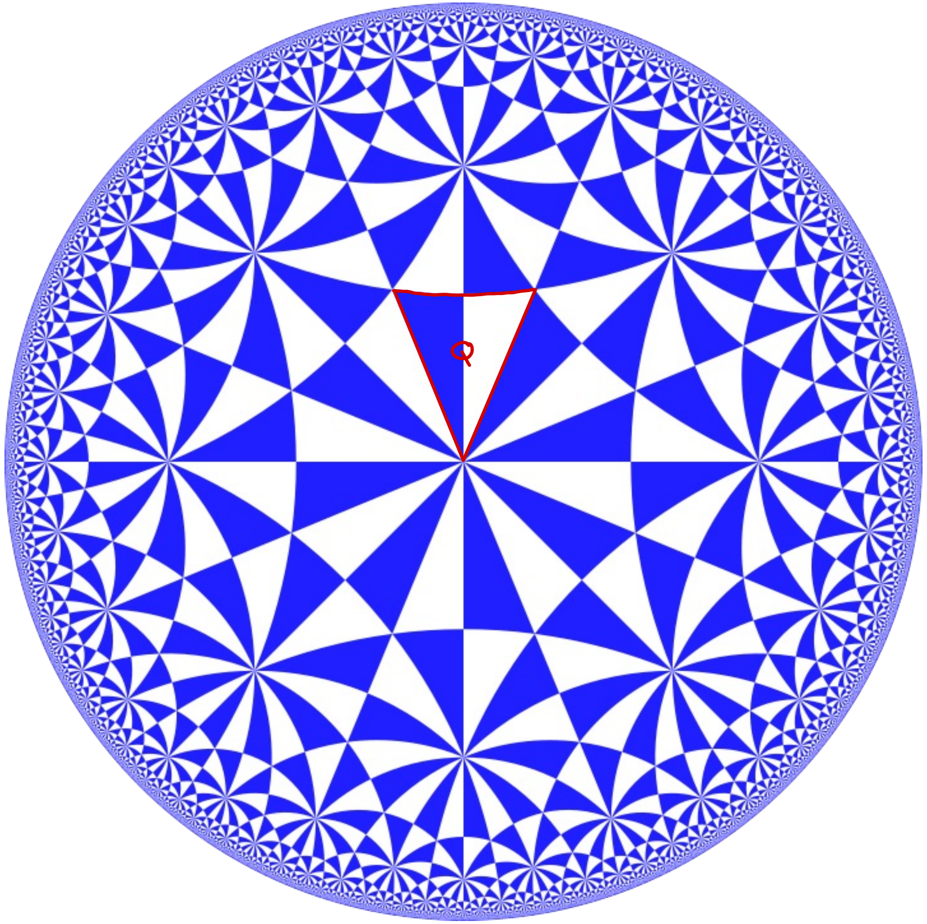
Furthermore,

$$\langle x, y, z \mid x^p = y^q = z^r = xyz = id \rangle$$

is a presentation for $T(p,q,r)$.

Fundamental domain



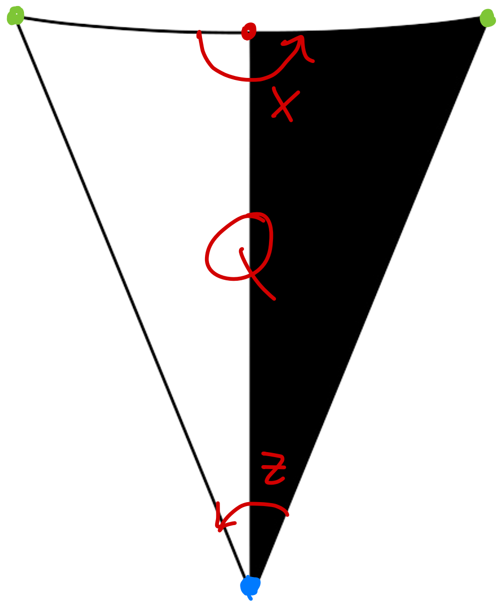


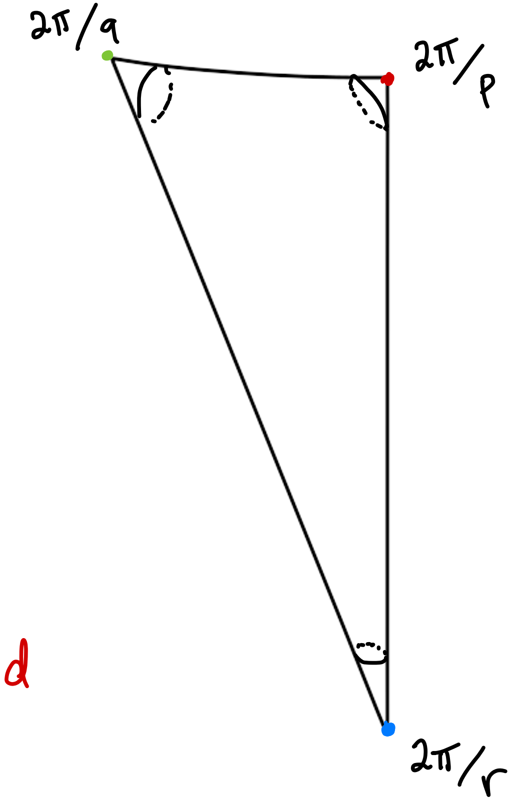
Quotient space

$$\mathcal{O}(p,q,r) := \mathbb{H}^2 / T(p,q,r)$$

$$\cong Q / \text{side identifications}$$

$$\cong$$





(p, q, r) - orbifold

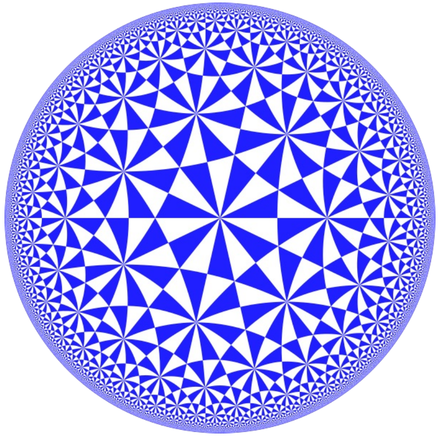
Triangle surfaces

Def^n A (p,q,r)-triangle surface is a surface of the form $X = \mathbb{H}^2/\Gamma$ where $\Gamma \triangleleft T(p,q,r)$ with Γ torsion-free and $[T:\Gamma] < \infty$.

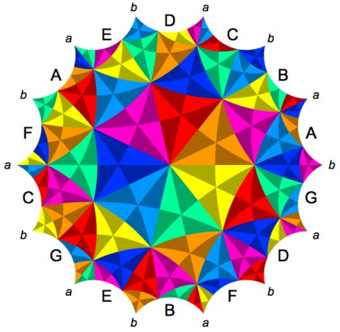
Then $G := T/\Gamma$ is a finite group which acts by orientation-preserving isometries on $X = \mathbb{H}^2/\Gamma$ and

$$X/G = \left(\mathbb{H}^2/\Gamma\right) \Big/ \left(T(p,q,r)/\Gamma\right) \cong \mathbb{H}^2 / T(p,q,r) = \mathcal{O}(p,q,r).$$

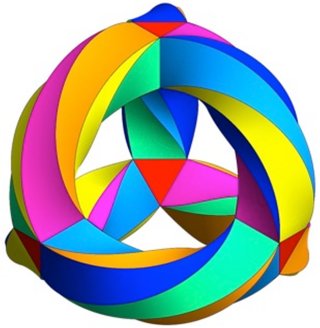
$$\begin{tikzcd} \mathbb{H}^2 \arrow[r, "/\Gamma"] \arrow[rr, bend right, "/T(p,q,r)"'] & X \arrow[r, "/G"] & O(p,q,r) \end{tikzcd}$$



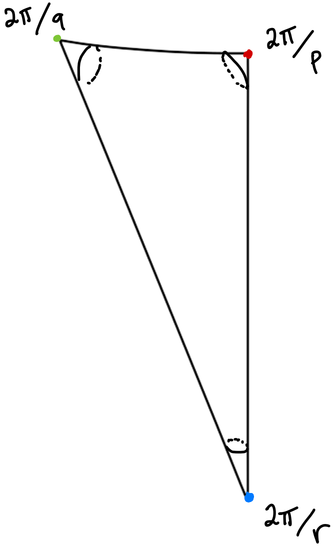
normal covering map



$||$



normal branched cover of finite degree



normal branched cover

triangle = quasiplatonic

Hurwitz surfaces

<u>Thm</u> (Hurwitz, 1893) If X is a closed hyperbolic surface of genus g (≥ 2), then $|\mathrm{Isom}^+(X)| \leq 84(g-1)$ with equality iff X is a $(2,3,7)$-triangle surface. (Hurwitz surface)

They exist in genus 3, 7, 14, 17, 118, ...

3 distinct ones

Properties of triangle surfaces

- X is a triangle surface of genus g

  $\iff$ X is a strict local maximizer of $|\mathrm{Isom}^+|$ in $\mathcal{M}_g$.

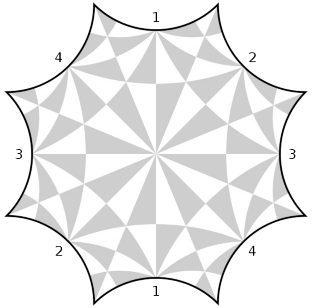
- Every triangle surface is a critical point of sys (on $\mathcal{T}_g$).

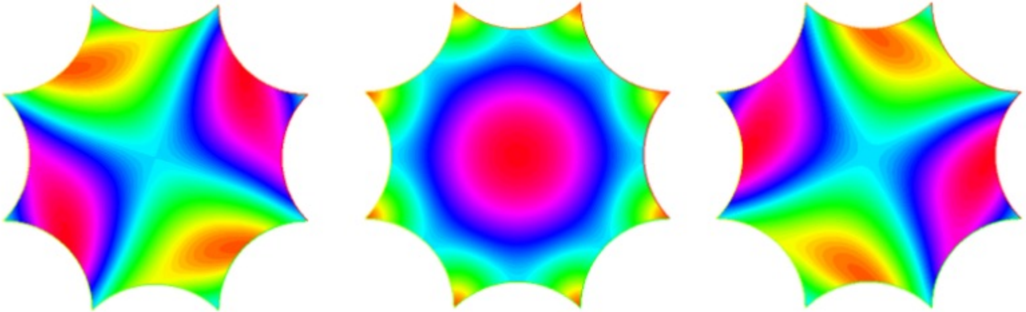
② Spectral data

The Bolza surface

$g = 2$

$(p,q,r) = (2,3,8)$





Thm (Jenni, '81) The Bolza surface satisfies

$$\lambda_1(B) \in [3.83, \; 3.85] \text{ and } m_1(B) = 3.$$

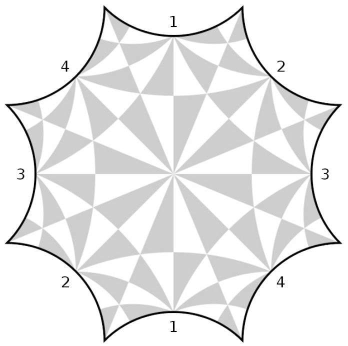
(Strohmaier-Uski, 2013) $\lambda_1(B) \approx 3.83888725884 2\ldots$

Cook (2018) discovered error in Jenni. Proved $3 \leq m_1(B) \leq 4$.

Thm (FB-Petri, '21) $m_1(B) = 3$. (Using the Cohn-Elkies method to prove $m_1(B) < 4$.)

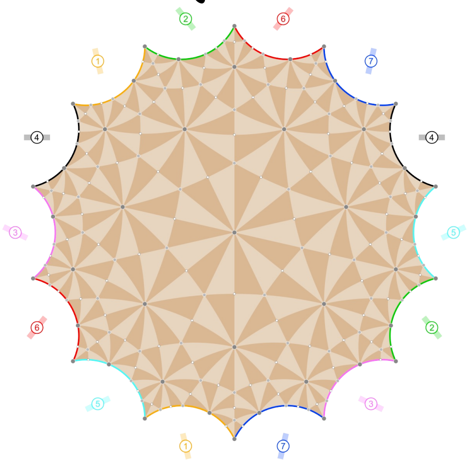
Cook's numerical data (Free Fem++)

Bolza, g = 2, (2,3,8)



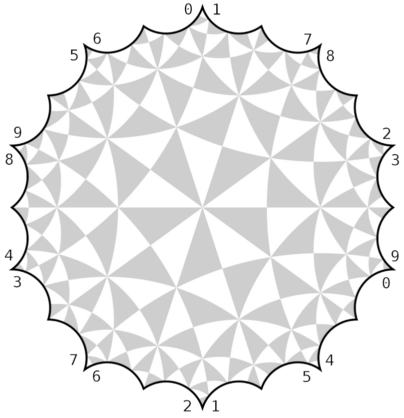
Eigenvalue	Numerical multiplicity
3.83618	3
5.35083	4
8.24401	2
14.7149	4
15.0386	3
18.645	3
20.5069	4
23.0545	1
28.0591	3
30.8081	4

Klein, g = 3, (2,3,7)



Eigenvalue	Numerical multiplicity
2.6767	8
6.61848	7
10.8637	6
12.1775	8
17.2397	7
21.9563	7
24.0649	8
25.9085	6
30.7817	6
36.4369	8

Bring, g = 4, (2,4,5)



Eigenvalue	Numerical multiplicity
1.91556	6
2.78954	5
5.89883	5
7.3379	5
8.26099	4
8.55338	6
13.2289	6
13.5082	4
15.1462	4
17.0459	6

Fricke-Macbeath (Hurwitz surface of genus 7)

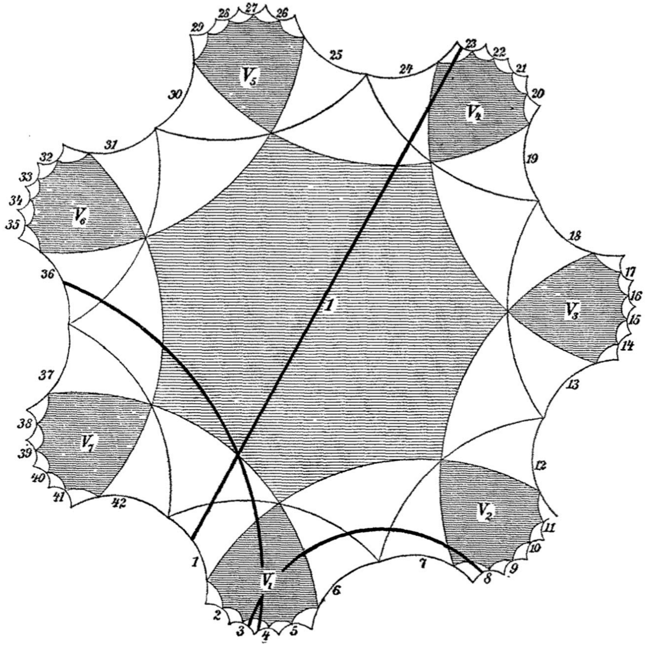


Fig. 3.

1—23
2—39
3— 8
4—36
5—25
6—16
7—29
8— 3
9—14
10—42
11—31
12—22
13—35
14— 9
15—20
16— 6
17—37
18—28
19—41
20—15
21—26
22—12
23— 1
24—34
25— 5
26—21
27—32
28—18
29— 7
30—40
31—11
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35—13
36— 4
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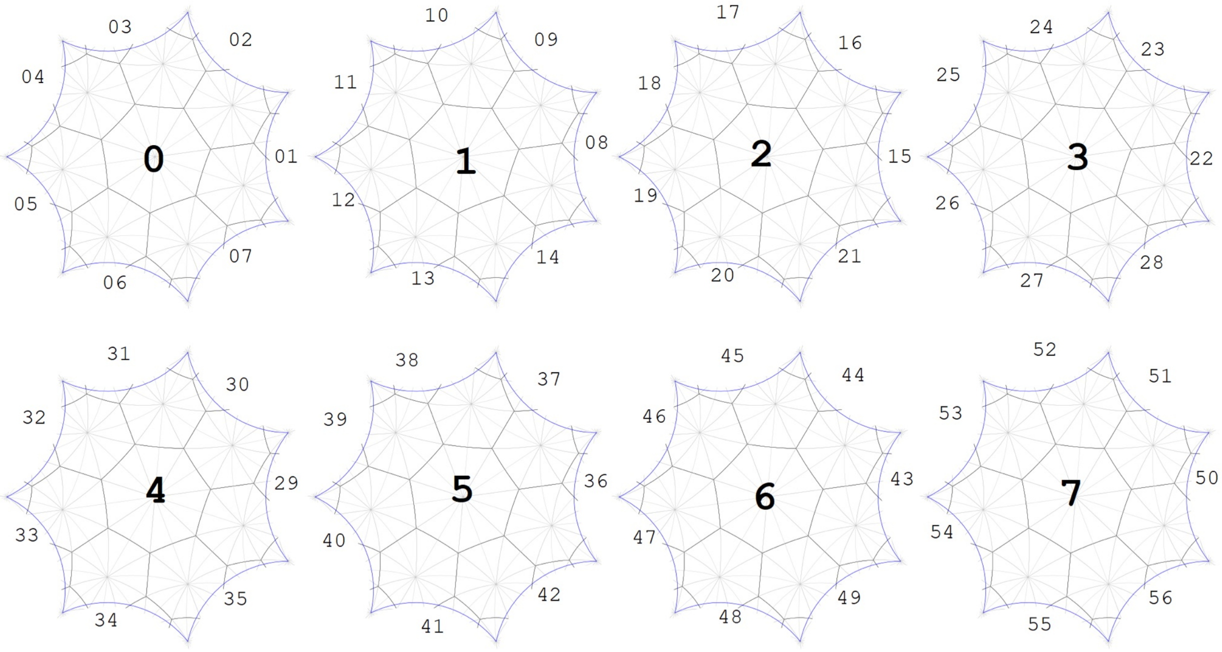
38—33
39— 2
40—30
41—19
42—10



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[01, 08]  [02, 15]  [03, 22]  [04, 29]  [05, 36]  [06, 43]  [07, 50]
[09, 34]  [10, 53]  [11, 17]  [12, 49]  [13, 37]  [14, 26]  [16, 41]
[18, 24]  [19, 56]  [20, 44]  [21, 33]  [23, 48]  [25, 31]  [27, 51]
[28, 40]  [30, 55]  [32, 38]  [35, 47]  [39, 45]  [42, 54]  [46, 52]
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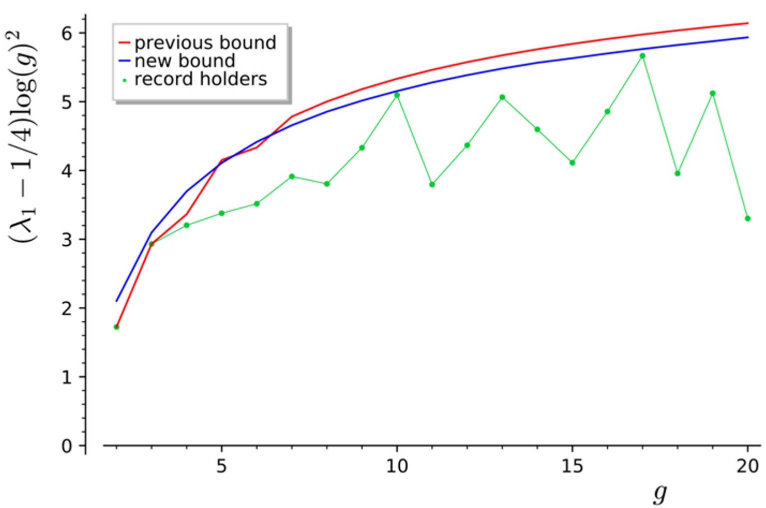
$\lambda_1(FM) \simeq 1.239$

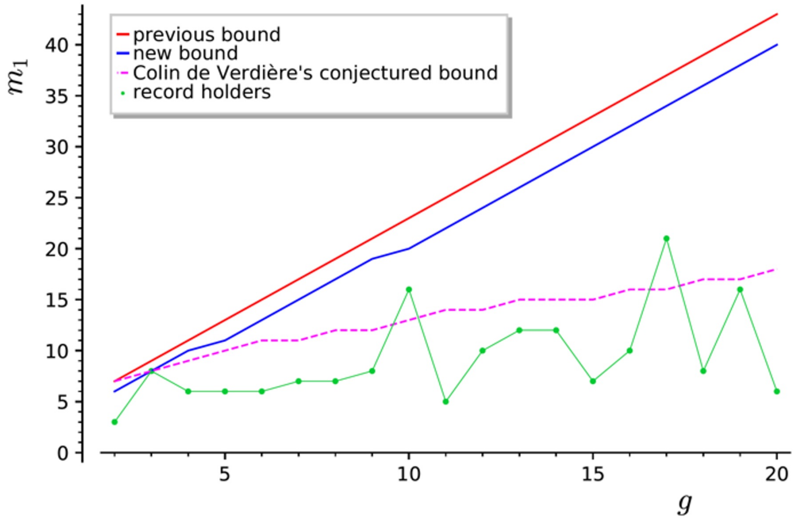
<u>Thm</u> (Pineault, Lee, '23) $m_1(FM) = 7$.

All triangle surfaces at once

If $X/G = O(p,q,r)$, then $X =$ $|G|$ copies of the quadrilateral Q glued along the Cayley graph of G wrt $\pi(x)$ & $\pi(z)$

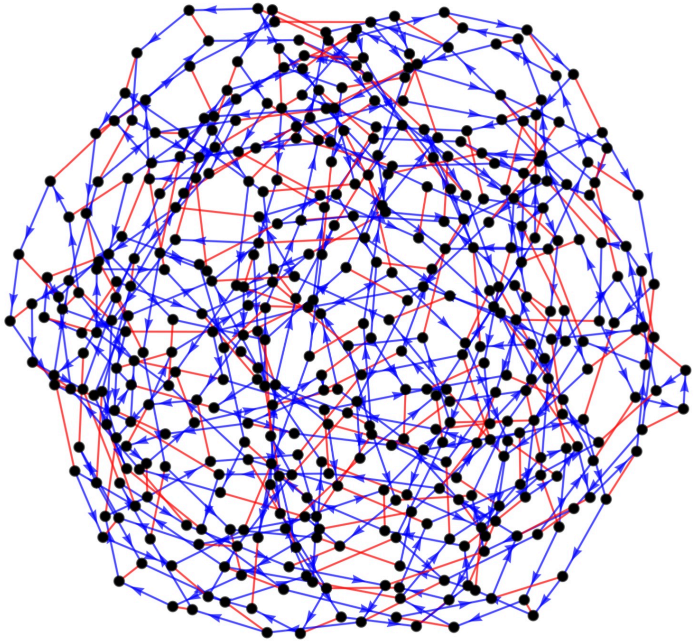
In summer 2023, Gruda-Mediavilla computed the bottom of spec(Δ_X) for all triangle surfaces of genus $2 \leq g \leq 20$. (tabulated by Conder)



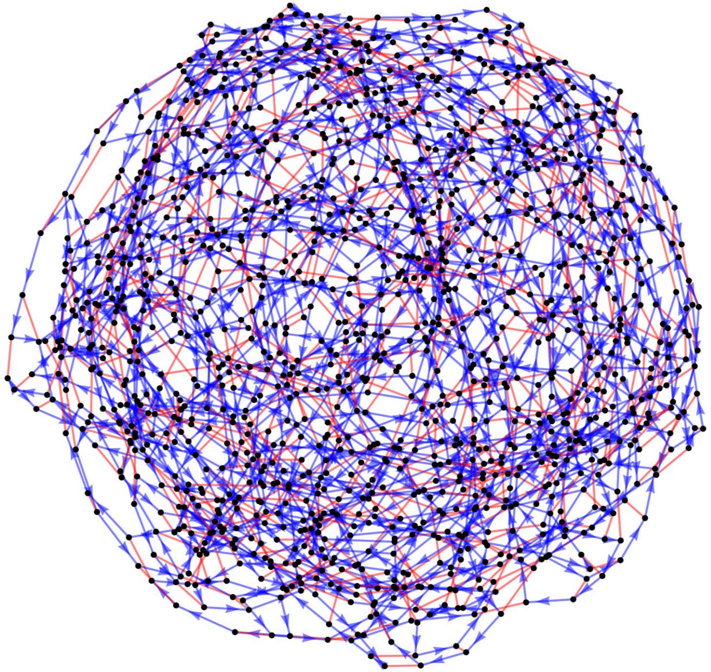


The counterexamples

$G_{10} \circlearrowright X_{10}$



$G_{17} \circlearrowright X_{17}$



Thm (FB-Gruda-Mediavilla-Petri-Pineault) $m_1(X_{10}) = 16$ and $m_1(X_{17}) = 21$.

③ Representation theory & multiplicity

If $G \leq \mathrm{Isom}(X)$ and E_λ is an eigenspace of the Laplacian, then

$$\begin{aligned} \rho : G &\to GL(E_\lambda) \\ g &\mapsto (f \mapsto f \circ g^{-1}) \end{aligned}$$ is a representation.

Maschke's thm $\Rightarrow$ $E_\lambda = \bigoplus_{j=1}^{K} V_j$ where $\rho(V_j) = V_j$ and $\rho|_{V_j}$ is irreducible.

$\Rightarrow \dim(E_\lambda) = \sum_{j=1}^{K} \dim(V_K)$

A finite group G has finitely many irreps (up to isomorphism) and they can be computed.

Strategy To prove a lower bound on $\dim(E_\lambda) = m(\lambda)$, it suffices to prove that irreps of small dimension cannot appear in E_λ.

How to rule out irreps?

trivial rep $\leftrightarrow$ G-invariant eigenfunctions on X

Can be ruled out if

① $\lambda \notin \operatorname{spec}(\Delta_{X/G})$

Jenni '81

Burger-Colbois '85

Sarnak-Xue '91

② if would contradict Courant's nodal domain theorem

Cook '18

FB-Petri '21

How about other irreps?

$X = \mathbb{H}^2/\Gamma$, $G \circlearrowright X$ by isometries,

so $X/G \cong \mathbb{H}^2/T$ for some $T \leq PSL(2,\mathbb{R})$ discrete.

$$T = \left\{ \text{all lifts} \begin{tikzcd} \mathbb{H}^2 \arrow[r, "\tilde{g}"] \arrow[d] & \mathbb{H}^2 \arrow[d] \\ X \arrow[r, "g"] & X \end{tikzcd} : g \in G \right\}$$

$\Rightarrow$ $\exists$ surjective homomorphism $\pi : T \to G$ with $\Gamma = \ker(\pi)$ hence $G \cong T/\Gamma$

Proposition (Cornelissen-Peyerimhoff, Sunada)

An irrep (ϕ, V) of G appears in the eigenspace E_λ of Δ_X $\iff$ $\exists$ nonzero ϕ-equivariant $F : X \to V$ s.t. $\Delta F = \lambda F$.

$\iff$ $\exists$ nonzero $(\phi \circ \pi)$-equivariant $\tilde{F} : \mathbb{H}^2 \to V$ s.t. $\Delta \tilde{F} = \lambda \tilde{F}$ ← twisted eigs

The twisted Selberg Trace Formula relates these ϕ-twisted eigenvalues to

- the conjugacy classes of primitive elements $\gamma \in \Gamma$
- the traces of $\phi \circ \pi(\gamma^n)$

via any admissible test function f & its Fourier transform.

$\Rightarrow$ Can use the Booker–Strömbergsson method to prove $\lambda_1(\phi) > U(g) \geq \lambda_1(X_g)$ for all but one irrep ϕ_g. $(g = 10, 17)$

$\Rightarrow$ Only ϕ_g can appear in $E_{\lambda_1(X_g)}$ $\Rightarrow m_1(X_g) \geq \dim(\phi_g) = \begin{cases} 16 \\ 21 \end{cases}$

BS plot for the irreps of G_{10} $(L = 7, \ n = 16)$